Quantum Optics MidTerm by Muhammad Sabieh Anwar

Total time allowed is 1 hour. Attempt all questions.

1. Consider the one-dimensional electric field inside a cavity,

$$E_x(z,t) = E_o \sin kz \, \sin \omega t. \tag{1}$$

Find the average electric field for the number state. **[5 marks]**

Note that we have access to he following relations,

$$\hat{q} = \left(\frac{\varepsilon_o \mathcal{V}}{2\omega^2}\right)^{1/2} E_o \sin \omega t \tag{2}$$

$$= \left(\frac{\hbar}{2\omega}\right)^{1/2} (\hat{a} + \hat{a}^\dagger). \tag{3}$$

2. This question deals with the pursuit of a suitable phase operator, analogous to the number operator $\hat{n} = \hat{a}^\dagger \hat{a}$ whose eigenvalues are the number of photons in the field. Consider the operator

$$\hat{F} = (\hat{n} + 1)^{-1/2} \hat{a}. \tag{4}$$

(a) What is the action of $\hat{F}$ on the number state? Mention the effect of operation on the vacuum state. You may like to recall that if $|a\rangle$ is an eigenstate of the Hermitian operator $\hat{A}$ with eigenvalue a, we also have,

$$f(\hat{A})|a\rangle = f(a)|a\rangle \tag{5}$$

for any continuous function f. **[5 marks]**

(b) Express $\hat{F}$ as a sum of outer products. (An outer product is of the form $|v\rangle\langle w|$.) You will use the orthonormality of the number states and the solution to the previous part. **[5 marks]**

(c) Show that the "phase-state"

$$|\phi\rangle = \sum_{n=0}^{\infty} e^{in\phi}\,|n\rangle, \tag{6}$$

where ϕ is a real angle is an eigenstate of $\hat{F}$. What is the corresponding eigenvalue? **[5 marks]**

(d) Consider an arbitrary photon state, which is a superposition of number states,

$$|\psi\rangle = \sum_{n=0}^{\infty} C_n|n\rangle, \tag{7}$$

where $\sum_{n=0}^{\infty}|C_n|^2 = 1$. Find the projection $\langle\phi|\psi\rangle$ and the so-called phase distribution function

$$P(\phi) = \frac{1}{2\pi}|\langle\phi|\psi\rangle|^2. \tag{8}$$

What is the distribution of phases $P(\phi)$ for a number state $|n\rangle$? **[5 marks]**

(e) Calculate the uncertainty in phase $\Delta\phi$ for a number state. **[5 marks]**

3. Consider no damping in a two-level atom which is excited with an off-resonant optical field. The electric field amplitude creates a Rabi frequency Ω_R which equals the detuning $\delta\omega$. What is the off-resonant nutation frequency? Write a unitary operator for this interaction. Assume the atom is initially in the ground state $|a\rangle$. What will be its quantum state after time t? Compared to an on-resonant field, will it take less time, or more time to go into an equal superposition $(|a\rangle + |b\rangle)/\sqrt{2}$ or will it never be able to go into this superposition? You may like to

use this theorem to find the unitary operator:

$$e^{i\theta\hat{A}} = \cos\theta\,\hat{\mathbf{1}} + i\sin\theta\hat{A}, \tag{9}$$

if $\hat{A}^2 = \mathbf{1}$. I expect that you sketch the approximate trajectory on the Bloch sphere. Can the equal superposition be created when $\delta\omega \gg \Omega_R$. Explain qualitatively (perhaps using your sketch on the Bloch sphere). **[15 marks]**

4. Suppose we have the superposition of coherent states,

$$|\psi\rangle = \frac{|\alpha\rangle + |\alpha e^{i2\pi/3}\rangle + |\alpha e^{i4\pi/3}\rangle}{\sqrt{3}}. \tag{10}$$

Note that in degrees, $2\pi/3 = 120°$ and $\cos 2\pi/3 = \cos 4\pi/3 = -1/2$. Find the probability distribution of the number of photons in this radiation field. **[10 marks]**