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Multimode fields — محمد صبيح النور

Wave equation: $\vec{\nabla}^2 \vec{A}(\vec{r},t) = \frac{1}{c^2} \frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2}$ ——— (1)

Trial solution : $\vec{A}(\vec{r},t) = \sum_{\vec{k},s} \hat{e}_{\vec{k},s} \left(A_{\vec{k},s}(t)\, e^{i\vec{k}\cdot\vec{r}} + A^*_{\vec{k},s}(t)\, e^{-i\vec{k}\cdot\vec{r}} \right)$ — (2)
(magnetic vector potential)

$$\boxed{s = \{s_1, s_2\}}$$

Putting the ansatz into (1), yields: $\forall\ \vec{k}, s$

$$\frac{d A_{\vec{k},s}(t)}{dt} = -\omega_k^2 A_{\vec{k},s}(t)$$

with the solut $A_{\vec{k},s}(t) = A_{\vec{k},s}\, e^{-i\omega_k t}$ ——— (3)

where $A_{\vec{k},s} = A_{\vec{k},s}(t=0)$.
shorthand

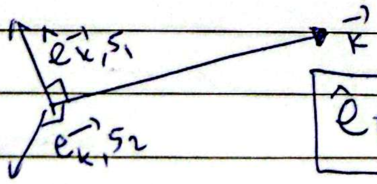
Furthermore, $A^*_{\vec{k},s}(t) = A^*_{\vec{k},s}\, e^{+i\omega_k t}$ ——— (4)

We want to find electric and magnetic fields from the complex amplitudes $\{ A_{\vec{k},s}(t),\ A^*_{\vec{k},s}(t) \}$ of the vector potential.

Coulomb gauge: $\vec{\nabla} \cdot \vec{A} = 0$ ——— (5)

translates to $\vec{k} \cdot \hat{e}_{\vec{k},s} = 0$ ——— (6a)

Right-handed triad →



$$\boxed{\hat{e}_{\vec{k},s} \cdot \hat{e}_{\vec{k},s'} = \delta_{s,s'}}$$ ——— (6b)

Also observe $\boxed{\hat{e}_{k,s_1} \times \hat{e}_{k,s_2} = \frac{\vec{k}}{|\vec{k}|} = \hat{k}}$ — (6c)

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Faraday's law: $\vec{E}(\vec{r},t) = -\frac{\partial \vec{A}(\vec{r},t)}{\partial t}$ — (7)

From (2) and (3) / (4), we have:

$$\vec{A}(\vec{r},t) = \sum_{\vec{k},s} \hat{e}_{\vec{k},s}\left(A_{\vec{k},s}\, e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + A^*_{\vec{k},s}\, e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}\right)$$

Hence from (7):

$$\boxed{\vec{E}(\vec{r},t) = -i\sum_{\vec{k},s} \hat{e}_{\vec{k},s}\, \omega_k \left(A_{\vec{k},s}\, e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} - A^*_{\vec{k},s}\, e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}\right)}$$ — (8)

This is still a classical field.

Turn to the magnetic field: $\vec{B} = \vec{\nabla} \times \vec{A}$

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)^T \quad \text{(def. of } \vec{\nabla}\text{)}$$

(^{T}: short hand for a vector in terms of its components)

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)^T$$

$$\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, -\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right), \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)^T$$

$$= i\left(k_y A_z - k_z A_y, -\left(k_x A_z - k_z A_x\right), k_x A_y - k_y A_x\right)^T$$

$$= i\,\vec{k} \times \vec{A}$$ — (9)

i. Fr... (...) ... (9)

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$$\vec{B}(\vec{r},t) = i\sum_{\vec{k},s}\left(\vec{k}\times e_{\vec{k},s}\right)\left(A_{\vec{k},s}(t)\,e^{i\vec{k}\cdot\vec{r}} - A^{*}_{\vec{k},s}(t)\,e^{-i\vec{k}\cdot\vec{r}}\right) \qquad (9)$$

take heed of this sign.

We have find electric and magnetic fields. Let's now find the energy we need:

$$U = \int_{V} dV\left(\frac{1}{2}\varepsilon_0\,\vec{E}(\vec{r},t)\cdot\vec{E}(\vec{r},t) + \frac{1}{2\mu_0}\vec{B}(\vec{r},t)\cdot\vec{B}(\vec{r},t)\right)$$

$$= U_E + U_B. \qquad (10)$$

$$\vec{E}\cdot\vec{E} = -\sum_{\vec{k},s}\hat{e}_{\vec{k},s}\,\omega_k\left(A_{\vec{k},s}(t)\,e^{i\vec{k}\cdot\vec{r}} - A^{*}_{\vec{k},s}(t)\,e^{-i\vec{k}\cdot\vec{r}}\right)$$

$$\cdot\sum_{\vec{k}',s'}\hat{e}_{\vec{k}',s'}\,\omega_{k'}\left(A_{\vec{k}',s'}(t)\,e^{+i\vec{k}'\cdot\vec{r}} - A^{*}_{\vec{k}',s'}(t)\,e^{-i\vec{k}'\cdot\vec{r}}\right)$$

$$= -\sum_{\substack{\vec{k},\vec{k}'\\ s,s'}}\left(\hat{e}_{\vec{k},s}\cdot\hat{e}_{\vec{k}',s'}\right)\omega_k\omega_{k'}\Big(A_{\vec{k},s}(t)\,A_{\vec{k}',s'}(t)\,e^{i(\vec{k}+\vec{k}')\cdot\vec{r}}$$

$$+ A^{*}_{\vec{k},s}(t)\,A^{*}_{\vec{k}',s'}(t)\,e^{-i(\vec{k}+\vec{k}')\cdot\vec{r}}$$

$$- A_{\vec{k},s}(t)\,A^{*}_{\vec{k}',s'}(t)\,e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} - A^{*}_{\vec{k},s}(t)\,A_{\vec{k}',s'}(t)\,e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}}\Big) \qquad (11)$$

From (6b), we can suppress the s' integral as only $s'=s$ terms matter in (11).

We also have $$\int_{\text{space}(V)} e^{i\vec{k}''\cdot\vec{r}}\, dV = V\delta_{\vec{k}'',\vec{0}} \quad \text{—— (12)}$$

∴ (11) becomes:

$$\int \vec{E}\cdot\vec{E} = -V\sum_{\vec{k},s}(\hat{e}_{\vec{k},s}\cdot\hat{e}_{\vec{k}',s})\,\omega_{\vec{k}}\,\omega_{\vec{k}'}\Big[\big\{A_{\vec{k},s}(t)A_{\vec{k}',s}(t) + A^*_{\vec{k},s}(t)A^*_{\vec{k}',s}(t)\big\}\delta_{\vec{k}',-\vec{k}} - \big\{A_{\vec{k},s}(t)A^*_{\vec{k}',s}(t) + A^*_{\vec{k},s}(t)A_{\vec{k}',s}(t)\big\}\delta_{\vec{k}',\vec{k}}\Big]$$

β is everything in the square brackets!

$$= -\sum_{\vec{k},s}(\hat{e}_{\vec{k},s}\cdot\hat{e}_{\vec{k}',s})\,\omega_{\vec{k}}\,\omega_{\vec{k}'}\,\beta \quad \text{—— (12b)}$$

$$= +V\sum_{\vec{k},s}\Big[\omega_k^2\,\big|A_{\vec{k},s}(t)\big|^2 + (\hat{e}_{\vec{k},s}\cdot\hat{e}_{-\hat{k},s})(\omega_{\vec{k}}\,\omega_{-\vec{k}})\big\{A_{\vec{k},s}(t)A_{-\vec{k},s}(t) + A^*_{\vec{k},s}(t)A^*_{-\vec{k},s}(t)\big\}\Big]$$

$$U_E = +\frac{\varepsilon_0}{2}V\sum_{\vec{k},s}\omega_k^2\big|A_{\vec{k},s}(t)\big|^2 - \frac{\varepsilon_0}{2}V\sum_{\vec{k},s}(\hat{e}_{\vec{k},s}\cdot\hat{e}_{-\hat{k},s})(\omega_{\vec{k}}\,\omega_{-\vec{k}})\big\{A_{\vec{k},s}(t)A_{-\vec{k},s}(t) + A^*_{\vec{k},s}(t)A^*_{-\vec{k},s}(t)\big\}$$

—— (13)

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$$\vec{B}\cdot\vec{B} = -\sum_{\vec{k},s}\left(\vec{k}\times\hat{e}_{\vec{k},s}\right)\left(A_{\vec{k},s}(t)e^{i\vec{k}\cdot\vec{r}} - A^*_{\vec{k},s}(t)\,e^{-i\vec{k}\cdot\vec{r}}\right)$$

$$\cdot\sum_{\vec{k}',s'}\left(\vec{k}'\times\hat{e}_{\vec{k}',s'}\right)\left(A_{\vec{k}',s'}(t)\,e^{i\vec{k}'\cdot\vec{r}} - A^*_{\vec{k}',s'}(t)\,e^{-i\vec{k}'\cdot\vec{r}}\right)$$

$$= -\sum_{\substack{\vec{k},\vec{k}'\\ s,s'}}\left(\vec{k}\times\hat{e}_{\vec{k},s}\right)\cdot\left(\vec{k}'\times\hat{e}_{\vec{k}',s'}\right)\left[\beta\right]$$ where β is defined : (12b).

(14)

We expand $\left(\vec{k}\times\hat{e}_{\vec{k},s}\right)\cdot\left(\vec{k}'\times\hat{e}_{\vec{k}',s'}\right)$ using the vector identity :

$$\left(\vec{A}\times\vec{B}\right)\cdot\left(\vec{C}\times\vec{D}\right) = \left(\vec{A}\cdot\vec{C}\right)\left(\vec{B}\cdot\vec{D}\right) - \left(\vec{A}\cdot\vec{D}\right)\left(\vec{B}\cdot\vec{C}\right)$$

thus

$$\left(\vec{k}\times\hat{e}_{\vec{k},s}\right)\cdot\left(\vec{k}'\times\hat{e}_{\vec{k}',s'}\right) = \left(\vec{k}\cdot\vec{k}'\right)\left(\hat{e}_{\vec{k},s}\cdot\hat{e}_{\vec{k}',s'}\right) - \left(\vec{k}\cdot\hat{e}_{\vec{k}',s'}\right)\cdot\left(\vec{k}'\cdot\hat{e}_{\vec{k},s}\right) \quad (15)$$

We have two kinds of term in β when we've integrated over V.

For terms which have $\delta_{\vec{k}',-\vec{k}}$, the vector identity (15) becomes

$$-k^2\left(\hat{e}_{\vec{k},s}\cdot\hat{e}_{-\vec{k},s'}\right) + \left(\vec{k}\cdot\hat{e}_{-\vec{k},s'}\right)\cdot\left(\vec{k}\cdot\hat{e}_{\vec{k},s}\right)$$

$$= -k^2\left(\hat{e}_{\vec{k},s}\cdot\hat{e}_{-\vec{k},s'}\right);$$ where the second term is zero because of (6a).

$= -k^2 \delta_{s,s'}$ (using 6b).

For terms containing $\delta_{\vec{k}',\vec{k}}$, the identity takes the form

$$k^2 (\hat{e}_{\vec{k},s} \cdot \hat{e}_{\vec{k},s'}) = k^2 \delta_{s,s'}$$

Hence

$$\int dV \vec{B} \cdot \vec{B} = + \sum_{\vec{k},s} k^2 (\hat{e}_{\vec{k},s} \cdot \hat{e}_{-\vec{k},s}) \left(A_{\vec{k},s}(t) A_{-\vec{k},s}(t) + A^*_{\vec{k},s}(t) A^*_{-\vec{k},s}(t) \right) + \sum_{\vec{k},s} k^2 |A_{\vec{k},s}(t)|^2 .$$

∴ we obtain

$$U_B = \frac{1}{2\mu_0} \int dV \vec{B} \cdot \vec{B}$$

$$= \frac{1}{2\mu_0} V \sum_{\vec{k},s} k^2 |A_{\vec{k},s}(t)|^2 + \frac{1}{2\mu_0} V \sum_{\vec{k},s} k^2 (\hat{e}_{\vec{k},s} \cdot \hat{e}_{-\vec{k},s}) \left(A_{\vec{k},s}(t) A_{-\vec{k},s}(t) + A^*_{\vec{k},s}(t) A^*_{-\vec{k},s}(t) \right)$$

As $k = \frac{\omega_k}{c}$

$$U_B = \frac{V}{2\mu_0 c^2} \sum_{\vec{k},s} \omega_k^2 |A_{\vec{k},s}(t)|^2 + \frac{1}{2\mu_0 c^2} \sum_{\vec{k},s} \omega_k^2 \left(A_{\vec{k},s}(t) A_{-\vec{k},s}(t) + A^*_{\vec{k},s}(t) A^*_{-\vec{k},s}(t) \right)$$

$$= \frac{\varepsilon_0}{2} V \sum_{\vec{k},s} \omega_k^2 |A_{\vec{k},s}(t)|^2 + \frac{\varepsilon_0}{2} V \sum_{\vec{k},s} \omega_k^2 \left(A_{\vec{k},s}(t) A_{-\vec{k},s}(t) + A^*_{\vec{k},s}(t) A^*_{-\vec{k},s}(t) \right) \quad \text{—— (16)}$$

Adding (13) and (16) gives

$$\boxed{U = \varepsilon_0 V \sum_{\vec{k},s} \omega_k^2 |A_{\vec{k},s}(t)|^2} \quad \text{—— (17)}$$

No B — I cast $A_{\vec{k},s}$ and $A^*_{\vec{k},s}$ in terms of $\hat{q}_{\vec{k},s}$ and $p_{\vec{k},s}$, which are position and momentum, thus:

$$A_{\vec{k},s} = \frac{1}{2\omega_k(\varepsilon_0 V)^{1/2}}\left(\omega_{\vec{k}}\, q_{\vec{k},s} + i\, p_{\vec{k},s}\right) \quad (18)$$

and

$$A^*_{\vec{k},s} = \frac{1}{2\omega_k(\varepsilon_0 V)^{1/2}}\left(\omega_{\vec{k}}\, q_{\vec{k},s} - i\, p_{\vec{k},s}\right) \quad (19)$$

Inserting (18) & (19) into (17)

$$U = \frac{1}{2}\sum_{\vec{k},s} \omega_k^2\, q^2_{\vec{k},s} + p^2_{\vec{k},s} \quad (20)$$

→ Convert to operators

$$\hat{H} = \frac{1}{2}\sum_{\vec{k},s} \omega_k^2\, \hat{q}^2_{\vec{k},s} + \hat{p}^2_{\vec{k},s} \quad (21)$$

And express in terms of ladder operators:

$$\hat{q}_{k,s} = \left(\frac{\hbar}{2\omega_k}\right)^{1/2}\left(\hat{a}_{\vec{k},s} + \hat{a}^{\dagger}_{\vec{k},s}\right)$$

$$\hat{p}_{k,s} = \frac{1}{i}\left(\frac{\hbar\omega_k}{2}\right)^{1/2}\left(\hat{a}_{\vec{k},s} - \hat{a}^{\dagger}_{\vec{k},s}\right)$$

$$\boxed{\hat{H} = \sum_{\vec{k},s} \hbar\omega_k\left(\hat{a}^{\dagger}_{\vec{k},s}\,\hat{a}_{\vec{k},s} + \frac{1}{2}\right)} \quad (22)$$