Quantum Optics: HW3 by Muhammad Sabieh Anwar

Please submit before 10 am, Thursday, 20 March 2025; only by the channel communicated

1. Answer the following questions relevant to the displacement operator.

   (a) Given the displacement operator $\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$, show that

   $$\hat{D}(\alpha)\ \hat{D}(-\alpha) = \mathbf{1}. \qquad (1)$$

   You may have to use $e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-[\hat{A},\hat{B}]/2}$ when $\left[A, [\hat{A},\hat{B}]\right] = \left[\hat{B}, [\hat{A},\hat{B}]\right] = 0$ and $\left[\hat{A},\hat{B}\right] \neq 0$.

   (b) Show using the result in part (a), that $\hat{D}(\alpha)$ is indeed unitary.

   (c) Show that the action of two successive displacement operators $\hat{D}(\alpha)$ followed by $\hat{D}(\beta)$, yields $D(\alpha+\beta)$ and $D(\alpha+\beta)|0\rangle - |\alpha+\beta\rangle$ up to a global phase.

2. This questions pertains the calculation of uncertainties in photon states.

   (a) Use the Heisenberg equation of motion $i\hbar\frac{d\hat{O}}{dt} = \left[\hat{O}, \hat{H}\right]$ to derive the time dependent forms of $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.

   (b) Use time dependent forms to find $\langle\hat{E}_x(t)\rangle$, $\langle\hat{E}_x^2(t)\rangle$ and $\Delta E_x(t)$ for number states.

   (c) Find ΔE_x for coherent states and show that the vacuum and coherent states have the same ΔE_x.

3. What is $\langle 0|\alpha\rangle$, where $|0\rangle$ is the vacuum state and $|\alpha\rangle$ is a coherent state?

4. This questions looks at an evolving coherent state.

   (a) Find the expectation values $\langle \hat{X}_1 \rangle$ and $\langle \hat{X}_2 \rangle$ for a coherent state $|\alpha e^{-\omega t}\rangle$.

   (b) Find the expectation value $\langle \hat{E}_x \rangle$ for this very state $|\alpha e^{-\omega t}\rangle$.

   (c) What if α is real? What is $\langle \hat{E}_x \rangle$ in this case?

5. A "cat" state is a superposition of coherent states. For example, consider $|\text{cat}\rangle = \frac{|\alpha\rangle + |-\alpha\rangle}{\sqrt{2}}$.

   (a) Show that $|\text{cat}\rangle$ is normalized only for large $|\alpha|$.

   (b) Find the probability distribution for the number of photons for $|\text{cat}\rangle$. What strange behavior do you observe?

6. A photon state is given by a superposition of numbers states.

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|n\rangle + e^{i\theta}|n+1\rangle), \tag{2}$$

   where θ is some real phase.

   (a) Starting with $\hat{E}_x(z,t) = E_\omega (\sin kz)(\hat{a} + \hat{a}^\dagger)$, establish the uncertainty principle between the observables $\hat{n}$ and $\hat{E}_x$. Find Δn, ΔE_x and $[\hat{n}, \hat{E}_x]$.

   (b) Show that this is *not* a minimum uncertainty state.

7. We need to show that the coherent state $|\alpha = \alpha' + i\alpha''\rangle$ does not undergo any dispersion. This means that with time, the wavefunction (in position) neither changes form nor its magnitude. This indicates robustness against decoherence. Answer the following questions.

(a) Show that the wavefunction is given by:

$$\langle q_o|\hbar\rangle = C_o\, e^{-\frac{1}{2q_o^2}(q-\sqrt{2}q_o\alpha')^2}, \tag{3}$$

where $q_o = \hbar/\omega$ and C_o is a coefficient that is independent of α.

(b) For this part of the question, assume that $\alpha'' = 0$. What does the wavefunction look like as a function of time? Does it oscillate between two extreme points? What are those turning points? Show that these turning points mark the intersection of the harmonic potential energy with the energy of the coherent state $\hbar\omega|\alpha|^2$. We are also assuming large intensities $|\alpha| \gg 0$.