HW Q1 ~~(a)~~. $D(\alpha)$ is unitary.

b) $$\left(D(\alpha)\right)^{\dagger} = \left(e^{-|\alpha|^2/2} e^{\alpha \hat{a}^{\dagger}} e^{-\alpha^* \hat{a}}\right)^{\dagger}$$

$$= e^{-|\alpha|^2/2} e^{-\alpha \hat{a}^{\dagger}} e^{\alpha^* \hat{a}} = D(-\alpha)$$

~~b)~~ a) where $$D(\alpha)D(-\alpha) = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^{\dagger}} e^{-\alpha^* \hat{a}} e^{-|\alpha|^2/2} e^{-\alpha \hat{a}^{\dagger}} e^{+\alpha^* \hat{a}}$$

$$= e^{-|\alpha|^2} \left(e^{\alpha \hat{a}^{\dagger}} \underline{e^{-\alpha^* \hat{a}} e^{-\alpha \hat{a}^{\dagger}}} e^{+\alpha^* \hat{a}} \right)$$

~~$\alpha \hat{a}^{\dagger} - \alpha^* \hat{a}$~~,

$$= e^{-|\alpha|^2} e^{\underline{\alpha \hat{a}^{\dagger} - \alpha^* \hat{a}}} e^{\underline{-\alpha \hat{a}^{\dagger} + \alpha^* \hat{a}}}$$

Let $A = \alpha \hat{a}^{\dagger} - \alpha^{*} \hat{a}$, $B = -\alpha \hat{a}^{\dagger} + \alpha^{*} \hat{a} = -A$.

$$[A,B] = 0.$$

$$[A,[A,B]] = 0$$

Let's focus on middle term: $e^{-\alpha^{*}\hat{a}} e^{-\alpha \hat{a}^{\dagger}}$

Let $A = -\alpha^{*}\hat{a}$, $B = -\alpha \hat{a}^{\dagger}$.

$$[A,B] = |\alpha|^2 [\hat{a}, \hat{a}^{\dagger}] = |\alpha|^2$$

$$[A,[A,B]] = [-\alpha^{*}\hat{a}, |\alpha|^2] = 0 = [B,[A,B]]$$

$$\therefore \quad e^{-\alpha^{*}\hat{a} - \alpha \hat{a}^{\dagger}} = e^{-\alpha^{*}\hat{a}} e^{-\alpha \hat{a}^{\dagger}} e^{-|\alpha|^2/2}$$

$$e^{A+B} = e^{A} . e^{B} e^{-[A,B]/2}$$

$$e^{B+A} = e^{B} e^{A} e^{-[B,A]/2}$$

$$e^{A} e^{B} e^{-[A,B]/2} = e^{B} e^{A} e^{-[B,A]/2}$$

$$e^{A} e^{B} = e^{B} e^{A} e^{-\frac{[B,A]}{2}} e^{+\frac{[A,B]}{2}}$$

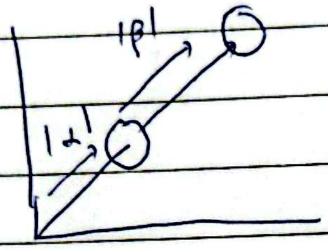
$$= e^{B} e^{A} e^{-[B,A]\cancel{/2}}.$$

$$\therefore \quad e^{-\alpha^{*}\hat{a}} e^{-\alpha \hat{a}^{\dagger}} = e^{-\alpha \hat{a}^{\dagger}} e^{-\alpha^{*}\hat{a}} e^{-[-\alpha\hat{a}^{\dagger}, -\alpha^{*}\hat{a}]\cancel{/2}}$$

Now $-[\alpha \hat{a}^{\dagger}, \alpha^{*}\hat{a}] = -|\alpha|^2 [\hat{a}^{\dagger}, \hat{a}] = |\alpha|^2$.

$$\therefore \quad D(\alpha) D(-\alpha) = e^{-|\alpha|^2} e^{+|\alpha|^2} \left(e^{\alpha \hat{a}^{\dagger}} e^{-\alpha \hat{a}^{\dagger}} e^{-\alpha^{*}\hat{a}} e^{+\alpha^{*}\hat{a}} \right) = \hat{1}.$$

(6) $\quad D(\beta)\, D(\alpha) \, |0\rangle = D(\alpha+\beta)|0\rangle$



$$D(\beta)\, D(\alpha) = e^{-|\beta|^2/2}\, e^{\beta\hat{a}^\dagger}\, e^{-\beta^*\hat{a}}\, e^{-|\alpha|^2/2}\, e^{\alpha\hat{a}^\dagger}\, e^{-\alpha^*\hat{a}}$$

$$= e^{-\frac{|\alpha|^2+|\beta|^2}{2}}\, e^{\beta\hat{a}^\dagger}\, e^{-\beta^*\hat{a}}\, e^{\alpha\hat{a}^\dagger}\, e^{-\alpha^*\hat{a}}$$

$$= e^{-\frac{(|\alpha|^2+|\beta|^2)}{2}}\, e^{\beta\hat{a}^\dagger}\, e^{\alpha\hat{a}^\dagger}\, e^{-\beta^*\hat{a}}\, e^{-\alpha^*\hat{a}}\, e^{-[\alpha\hat{a}^\dagger, \ldots}$$

$$= e^{-\left(\frac{|\alpha|^2+|\beta|^2}{2}\right)}\, e^{(\alpha+\beta)\hat{a}^\dagger}\, e^{-(\alpha^*+\beta^*)\hat{a}}\, e^{-[\alpha\hat{a}^\dagger, -\beta^*\hat{a}]}$$

Now $-\ [\alpha\hat{a}^\dagger, -\beta^*\hat{a}] = \alpha\beta^*\, [\hat{a}^\dagger, \hat{a}] = -\alpha\beta^*$

$$\therefore\ D(\beta)\, D(\alpha) = e^{-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} - \alpha\beta^*}\, e^{(\alpha+\beta)\hat{a}^\dagger}\, e^{-(\alpha^*+\beta^*)\hat{a}}$$

$$\rightarrow\ D(\beta)\, D(\alpha)\, |0\rangle = D(\beta)\, |\alpha\rangle$$

$$= D(\beta)\, e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha)^n}{\sqrt{n!}} |n\rangle$$

$$= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha)^n}{\sqrt{n!}}\, e^{(\beta\hat{a}^\dagger - \beta^*\hat{a})}\, |n\rangle$$

$$= e^{-|\alpha^2|/2}\, e^{-|\beta|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha)^n}{\sqrt{n!}}\, e^{\beta\hat{a}^\dagger}\, e^{-\beta^*\hat{a}}\, |n\rangle$$

$$e^{-\beta^* \hat{a}} = \sum_{p=0}^{\infty} \frac{(-\beta^* \hat{a})^p}{p!}$$

$$\boxed{|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle}$$

$$\therefore D(\beta)|\alpha\rangle = e^{-|\alpha|^2/2} e^{-|\beta|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha)^n}{\sqrt{n!}} \frac{e^{\beta \hat{a}^\dagger} e^{-\beta^* \hat{a}}}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle$$

<u>$D(\alpha+\beta)$</u>

$$e^{-\beta^* \hat{a}} |n\rangle = \left(\sum_{p=0}^{\infty} \frac{(-\beta^* \hat{a})^p}{p!} \right) |n\rangle$$

Get's complicated

$$\cancel{-\frac{\beta^* \sqrt{n}}{p!}}$$

$$\boxed{D(\alpha+\beta) = \cancel{e^{-\frac{|\alpha+\beta|^2}{2}}}\; e^{+(\alpha+\beta)\hat{a}^\dagger - (\alpha+\beta)^* \hat{a}}}$$ ← Show $D(\alpha+\beta) = D(\alpha) D(\beta)$.

$$[(\alpha+\beta)\hat{a}^\dagger, -(\alpha+\beta)^* \hat{a}] = -|(\alpha+\beta)|^2 [\hat{a}^\dagger, \hat{a}] = |\alpha+\beta|^2 .$$

Now $e^{\hat{A}+\hat{B}} = e^{A} e^{B} e^{-[A,B]/2}$.

Check. $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$ ✓ Yes.

$$\cancel{\therefore D(\alpha+\beta) = e^{-\frac{|\alpha+\beta|^2}{2}} e^{\frac{|\alpha+\beta|^2}{2}} e^{(\alpha+\beta)\hat{a}^\dagger} e^{-(\alpha+\beta)^* \hat{a}}}$$

$$D(\alpha)\, D(\beta) = \cancel{e^{-\frac{|\alpha|^2+|\beta|^2}{2}}}\; e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}\; e^{\beta \hat{a}^\dagger - \beta^* \hat{a}}$$

$$A = \alpha \hat{a}^\dagger - \alpha^* \hat{a}, \qquad B = \beta \hat{a}^\dagger - \beta^* \hat{a}$$

$$[A, B] = [\alpha \hat{a}^\dagger - \alpha^* \hat{a},\ \beta \hat{a}^\dagger - \beta^* \hat{a}]$$

$$= \alpha [\hat{a}^\dagger,\ \beta \hat{a}^\dagger - \beta^* \hat{a}] - \alpha^* [\hat{a},\ \beta \hat{a}^\dagger - \beta^* \hat{a}]$$

$$= \alpha(-\beta^*)(-1) - \alpha^* \beta = \alpha\beta^* - \alpha^*\beta = 2i\,\mathrm{Im}(\alpha\beta^*)$$

$$\neq 0.$$

$$[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0.$$

$$D(\alpha)D(\beta) = e^{i\,\mathrm{Im}(\alpha\beta^*)} D(\alpha+\beta)$$

global phase.

Q4 (a)

$$\langle \hat{X}_1 \rangle = \frac{1}{2} \langle \alpha e^{-i\omega t} | (\hat{a} + \hat{a}^{\dagger}) | \alpha e^{-i\omega t} \rangle$$

$$\hat{a} | \alpha e^{-i\omega t} \rangle = \alpha e^{-i\omega t} | \alpha e^{-i\omega t} \rangle$$

$$\langle \alpha e^{-i\omega t} | \hat{a}^{\dagger} = \alpha^{*} e^{i\omega t} \langle \alpha e^{-i\omega t} |$$

$$\langle \hat{X}_1 \rangle = \frac{1}{2} \left(\alpha e^{-i\omega t} + \alpha^{*} e^{i\omega t} \right) = \frac{1}{2} \cdot 2 \, \mathrm{Re} \left(\alpha e^{-i\omega t} \right)$$

$$= \mathrm{Re} \left(\alpha e^{-i\omega t} \right) = \mathrm{Re} \left\{ \left(\alpha' \cancel{\cos\omega t} + i\alpha'' \right) \left(\cos\omega t - i \sin\omega t \right) \right.$$

$$\boxed{\langle \hat{X}_1 \rangle = \alpha' \cos\omega t + \alpha'' \sin\omega t}$$

$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

$\langle(\hat{a}+\hat{a}^\dagger)\rangle_\alpha = \alpha + \alpha^*$

$\langle\alpha|\left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{+i\omega t}\right)|\alpha\rangle = \alpha e^{-i\omega t} + \alpha^* e^{+i\omega t}$.

$= 2\mathrm{Re}\left(\alpha e^{-i\omega t}\right)$

$= 2\left(\alpha' \cos\omega t \; \cancel{+} \; \alpha'' \sin\omega t\right)$.

$(\alpha' + i\alpha'')(\cos\omega t - i\sin\omega t)$

$= \underline{\alpha' \cos\omega t + \alpha'' \sin\omega t}$.

$i\hbar \frac{d\hat{a}}{dt} = \cancel{[H, a]} \; [\hat{a}, \hat{H}]$

COMPARE WITH THIS.

$$\boxed{\frac{d\hat{a}}{dt} = \frac{1}{i\hbar}[\hat{a}, H] = \frac{i}{\hbar}[\hat{H}, \hat{a}]}$$

This is correct.

$\hbar\omega[\hat{a}^\dagger\hat{a}, \hat{a}] = \hbar\omega\left(\hat{a}^\dagger\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{a}\right)$

$= \hbar\omega [\hat{a}^\dagger, \hat{a}]\hat{a} = -\hbar\omega\hat{a}$

$\frac{d\hat{a}}{dt} = -i\frac{\hbar\omega}{\hbar}\hat{a} \Rightarrow \frac{d\hat{a}}{dt} + i\omega\hat{a} = 0$

$e^{i\omega t}\frac{da}{dt} + i\omega e^{i\omega t} a = 0$

$\frac{d}{dt}(e^{i\omega t} a) = 0$

$e^{i\omega t} a = K$

$a = K e^{-i\omega t}$

$\hat{a}(t) = \hat{a}\, e^{-i\omega t}$

Q4 (b)

$$\langle E_x \rangle = E_0 \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \sin kz \, \langle \alpha e^{-i\omega t} | (\hat{a} + \hat{a}^\dagger) | \alpha e^{-i\omega t} \rangle$$

$$= \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \sin kz \left(\alpha e^{-i\omega t} + \alpha^* e^{i\omega t} \right)$$

$$= \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \sin kz \left(2 \,\mathrm{Re} \left(\alpha e^{-i\omega t} \right) \right)$$

$$= 2 \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \sin kz \left(\alpha' \cos\omega t + \alpha'' \sin\omega t \right)$$

If α is real $\left(2 \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \sin kz \, \alpha \cos\omega t \right)$.

Q5. (a) $|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\alpha\rangle + |-\alpha\rangle \right)$

$$\langle \psi | \psi \rangle = \frac{1}{2} \left(\langle \alpha | + \langle -\alpha | \right) \left(|\alpha\rangle + |-\alpha\rangle \right)$$

$$= \frac{1}{2} \left(\langle \alpha | \alpha \rangle + \langle -\alpha | -\alpha \rangle + \langle \alpha | -\alpha \rangle + \langle -\alpha | \alpha \rangle \right)$$

$$\langle \alpha | \alpha \rangle = \left(e^{-|\alpha|^2/2} \sum \frac{(\alpha^*)^m \langle m |}{\sqrt{m!}} \right) e^{-|\alpha|^2/2} \sum \frac{(\alpha)^n}{\sqrt{n!}} |n\rangle$$

$$= e^{-|\alpha|^2} \sum_{\cancel{\sqrt{m!}\sqrt{n!}}\, m,n} \frac{1}{\sqrt{m!}\sqrt{n!}} (\alpha^*)^m (\alpha)^n \langle m | n \rangle$$

$$= e^{-|\alpha|^2} \sum_n \frac{1}{n!} (|\alpha|)^{2n}$$

$$= e^{-|\alpha|^2} \sum_n \frac{1}{n!} (|\alpha|^2)^n = 1.$$

$\boxed{\text{Similarly:}}$

$$\langle -\alpha | -\alpha \rangle = e^{-|\alpha|^2} \sum_{m,n} \frac{1}{\sqrt{m!}\sqrt{n!}} (-\alpha^*)^m (-\alpha)^n \langle m | n \rangle$$

$$= e^{-|\alpha|^2} \sum_{m,n} \frac{1}{\sqrt{m!}\sqrt{n!}} (-1)^{m+n} |\alpha|^{2\,m+n} \langle m | n \rangle$$

$$= 1.$$

$$\langle -\alpha | \alpha \rangle = e^{-|\alpha|^2/2} e^{-|\alpha|^2/2} \sum_m \frac{(-\alpha^*)^m \langle m|}{\sqrt{m!}} \sum_n \frac{(\alpha)^n |n\rangle}{\sqrt{n!}}$$

$$= e^{-|\alpha|^2} \sum_{m,n} \frac{(-1)^m (\alpha^*)^m (\alpha)^n \langle m | n \rangle}{\sqrt{m!}\sqrt{n!}} \rightarrow \delta_{m,n}$$

$$= e^{-|\alpha|^2} \sum_{m,n} \frac{(-1)^n (\alpha^*)^n (\alpha)^n}{n!}$$

$$= e^{-|\alpha|^2} \sum_n \frac{(-1)^n (|\alpha|^2)^n}{n!}$$

$$= e^{-|\alpha|^2} \left[\left(1 + \frac{|\alpha|^4}{2!} + \frac{|\alpha|^8}{4!} + \frac{|\alpha|^{12}}{6!} + \dots \right) - \left(|\alpha|^2 + \frac{|\alpha|^6}{3!} + \frac{|\alpha|^{10}}{5!} + \dots \right) \right]$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{4!} - \frac{x^6}{\dots} + \dots$$

$$= e^{-|\alpha|^2} e^{-|\alpha|^2}$$

$$= e^{-2|\alpha|^2} \rightarrow 0 \text{ as } |\alpha| \text{ is large.}$$

So $\langle\psi|\psi\rangle = 1 + e^{-2|\alpha|^2}$. Cat state is not normalized.

(b) $\langle n|\psi\rangle = \frac{1}{\sqrt{2}}\left(\langle n|\alpha\rangle + \langle n|-\alpha\rangle\right)$

Now $\langle n|\alpha\rangle = e^{-|\alpha|^2/2}\langle n| \sum_{m=0}^{\infty} \frac{(\alpha)^m}{\sqrt{m!}}|m\rangle$

$$= e^{-|\alpha|^2/2}\frac{(\alpha)^n}{\sqrt{n!}} \quad \cancel{= \frac{\bar{n}^{n} e^{-\bar{n}}}{\sqrt{n!}}}$$

$$|\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2/1}\frac{(\alpha)^{2n}}{n!} = \frac{\bar{n}^n e^{-\bar{n}}}{n!}.$$

Now $\langle n|-\alpha\rangle = e^{-|\alpha|^2/2}\frac{(-\alpha)^n}{\sqrt{n!}}$

$$\langle n|\psi\rangle = \frac{1}{\sqrt{2}}\frac{e^{-|\alpha|^2/2}}{\sqrt{n!}}\left((\alpha)^n + (-\alpha)^n\right)$$

$$|\langle n|\psi\rangle|^2 = \frac{1}{2}\frac{e^{-|\alpha|^2}}{n!}\left((\alpha^*)^n + (-\alpha^*)^n\right)\left((\alpha)^n + (-\alpha)^n\right)$$

$$= \frac{1}{2} \frac{e^{-|\alpha|^2}}{n!} \left((|\alpha|^2)^n + (-|\alpha|^2)^n + (-|\alpha|^2)^n + (|\alpha|^2)^n \right)$$

$$= \frac{e^{-|\alpha|^2}}{n!} \left(|\alpha|^{2n} + (-1)^n |\alpha|^{2n} \right)$$

$$= \begin{cases} 2 \frac{e^{-|\alpha|^2}}{n!} (|\alpha|^2)^n, & n \text{ even.} \\ 0, & n = \text{odd} \end{cases}$$

If n is odd, no photons exist.

Photons exist only for n even.

→ Poissonian for even.