PHY 312 **Due by:**
April 14, 2025 **Homework 4** April 24, 11 am

Problem 1

Let $\mathbf{J}_1$ and $\mathbf{J}_2$ be the respective angular momenta of the individual components of a two-component system. The total system has angular momentum $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$. Show that:

(a) $\mathbf{J}_1 \cdot \mathbf{J}_2 = \frac{1}{2}(J_{1+}J_{2-} + J_{1-}J_{2+}) + J_{1z}J_{2z}$

(b) $J^2 = J_1^2 + J_2^2 + 2J_{1z}J_{2z} + (J_{1+}J_{2-} + J_{1-}J_{2+})$

Problem 2

Consider the case where $j = \frac{3}{2}$.

(a) Find the matrices representing the operators $\hat{J}^2, \hat{J}_z, \hat{J}_\pm, \hat{J}_x$ and $\hat{J}_y$. Mention the basis which is used for representation.

(b) Find the joint eigenstates of $\hat{J}^2$ and $\hat{J}_z$, and verify that they form an orthonormal and complete basis.

(c) Use the matrices of $\hat{J}_x$, $\hat{J}_y$ and $\hat{J}_z$ to calculate $[\hat{J}_x, \hat{J}_y]$, $[\hat{J}_y, \hat{J}_z]$, and $[\hat{J}_z, \hat{J}_x]$.

(d) If the Hamiltonian for a spin-$\frac{3}{2}$ particle is given by

$$\hat{H} = \omega_0 \hat{S}_z$$

and at time $t = 0$, $|\psi(0)\rangle = |3/2, 3/2\rangle$, determine the probability that the particle is in the state $|3/2, -3/2\rangle$ at time t. Evaluate this probability when $t = \pi/\omega_0$ and explain your result.

Problem 3

Consider a particle of total angular momentum $j = 1$. Find the matrix for the component of $\hat{J}$ along a unit vector with arbitrary direction $\hat{n}$. Find its eigenvalues and eigenvectors.

Problem 4

Consider the operator

$$\hat{A} = \frac{1}{2}(\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x)$$

Calculate the expectation value of $\hat{A}$ and $\hat{A}^2$ with respect to the state $|j, m\rangle$.

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Problem 5

Consider a system of total angular momentum $j = 1$. We are interested here in the measurement of $\hat{J}_y$. Its matrix in the simultaneous eigenbasis of $\hat{J}^2$ and $\hat{J}_z$ is given by

$$\hat{J}_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

(a) What are the possible values we will obtain when measuring $\hat{J}_y$?

(b) Calculate $\langle \hat{J}_z \rangle$, $\langle \hat{J}_z^2 \rangle$, and ΔJ_z if the system is in the state $j_y = \hbar$.

Problem 6

A spin-1 particle with a magnetic moment $\boldsymbol{\mu} = (gq/2m)\mathbf{S}$ is situated in a magnetic field $\mathbf{B} = B_0\hat{k}$ in the z-direction. At time $t = 0$, the particle is in the state

$$|1,1\rangle_y = \frac{1}{2}|1,1\rangle + i\frac{\sqrt{2}}{2}|1,0\rangle - \frac{1}{2}|1,-1\rangle$$

with $S_y = \hbar$. Here, the unsubscripted kets represent eigenstates of $\hat{S}^2$ and $\hat{S}_z$. Determine the state of the particle at time t. Calculate how the expectation values $\langle S_y \rangle$ and $\langle S_z \rangle$ vary in time.

Problem 7

Find the matrix representations of S_x, S_y, S_z and S_n for spin 1/2 system.

Problem 8

Consider a spin-$\frac{1}{2}$ particle with a magnetic moment.

(a) At time $t = 0$, the observable S_x is measured, with the result $\frac{\hbar}{2}$. What is the state vector $|\psi(t = 0)\rangle$ immediately after the measurement?

(b) Immediately after the measurement, a magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}(H = \omega_0 S_z)$ is applied and the particle is allowed to evolve for a time T. What is the state of the system at time $t = T$?

(c) At $t = T$, the magnetic field is very rapidly changed to $\mathbf{B} = B_0\hat{\mathbf{y}}(H = \omega_0 S_y)$. After another time interval T, a measurement of S_x is carried out once more. What is the probability that a value $\frac{\hbar}{2}$ is found?

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Problem 9

A proton is placed in a uniform magnetic field pointing in the z-direction:

$$\vec{B} = B\hat{z}$$

(a) What is the precession of the spin if the initial spin state lies in the x-y plane, at an angle ϕ with respect to the x-axis? Find the time-evolved state $\Psi(t)$, Draw the time evolution of the state on the Bloch sphere for different values of ϕ.

(b) What are the time-dependent expectation values of the spin components $\langle S_x(t)\rangle$, $\langle S_y(t)\rangle$, and $\langle S_z(t)\rangle$?

Problem 10

Find the Fourier Transform of the following function, defined for $t > 0$, and plot its magnitude. Show that the result is a Lorentzian function.

$$f(t) = e^{i\omega_0 t - \frac{t}{T}}, \quad t > 0$$

Problem 11

Show that if $[\hat{A}, \hat{B}] = i\hat{C}$, the following sandwich theorem holds:

$$e^{-i\theta\hat{C}}\hat{A}e^{i\theta\hat{C}} = \cos\theta\,\hat{A} + \sin\theta\,\hat{B}$$

and that this helps explain the result $e^{-\frac{i\theta\hat{S}_z}{\hbar}}\hat{S}_x e^{\frac{i\theta\hat{S}_z}{\hbar}} = \hat{S}_x cos\theta + \hat{S}_y sin\theta$.

Problem 12

Starting from $|\alpha\rangle$ we desire to take the quantum state of the spin-1/2 particle through the following sequence

$$|\widetilde{\alpha}\rangle \rightarrow |\widetilde{x}\rangle \rightarrow |\widetilde{y}\rangle \rightarrow |-\widetilde{y}\rangle \rightarrow |\widetilde{x}\rangle.$$

Here the kets with tildes are states in the rotating frame. Draw a timing sequence of magnetic fields and phases that need to be applied to achieve this choreography of the spin vector. Specialists call this sequence a *spin echo*.

Problem 13

A magnetic field is pointing in the z-direction corresponding to a Hamiltonian $\omega_0 \hat{S}_z$. A spin-1 particle is placed inside the field. Its initial state, written in the eigenbasis of $\hat{S}_z$, is:

$$|\psi\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

Given:

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{\sqrt{2}i} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

(a) What is the state after a time t? Don't forget to mention your basis if you write a matrix representation.

(b) After time t, the component $\hat{S}_x$ is measured. What is the probability of obtaining zero as the measurement outcome?

Problem 14

The Hamiltonian of a three-level system is

$$\begin{pmatrix} E_0 & 0 & A \\ 0 & E_0 & 0 \\ A & 0 & E_0 \end{pmatrix}$$

when written in the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis in the same order. If the system is in the state $|3\rangle$ at time $t = 0$, how long will it take to convert to $|1\rangle$?$|2\rangle$?