Quantum Mechanics-2 MidTerm by Muhammad Sabieh Anwar

Total time allowed is 1 hour. Attempt both questions.

1. Consider an electron trapped in a 1 nm sized nanoparticle. Such a particle behaves like a quantum dot. The situation is well described by an infinite spherical well with

$$V(r) = \begin{cases} 0, & r \leq a \\ \infty & r > a. \end{cases}$$

   (a) Write the Hamiltonian in spherical coordinates. [**3**]

   Note $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$

   (b) Show that the radial part of the wave equation is given by [**8**]

$$-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left(\frac{l(l+1)\hbar^2}{2mr^2}+V(r)-E\right)R=0.$$

   You will use the differential form of the orbital angular momentum operator

$$\hat{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}\right).$$

   (c) Show that for $l = 0$, and for $r < a$, the radial part can also be written as [**4**]

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}(rR_{n,0}(r))-E(rR_{n,0}(r))=0.$$

   (d) Set $rR_{n,0}(r) = u_{n,0}(r)$ and solve the equation to find the radial wave functions for the quantum dot for $l = 0$. Use the appropriate boundary conditions (wavefunction disappears at $r \to 0$; what happens at $r \to \infty$?). [**8**]

(e) Find the normalized wavefunction $\psi_{n,0}(r,\theta,\phi)$, which is an eigenstate of the Hamiltonian by noting that [**7**]

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}|Y_{m_l}^{l}(\theta,\phi)|^2 \sin\theta d\theta d\phi = 1$$

and

$$Y_0^0(\theta,\phi) = \sqrt{\frac{1}{4\pi}}.$$

2. For the simple harmonic oscillator, for which

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega(\hat{a}^\dagger\,\hat{a} + \frac{1}{2})$$

where the position operator can be expressed as,

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger).$$

The Hamiltonian has eigenstates $|n\rangle$ such that $\hat{a}^\dagger\,\hat{a}|n\rangle = n|n\rangle$ where n is a non-zero negative integer and $|n\rangle$ is called the number state. We also know that $[\hat{a}^\dagger, \hat{a}] = 1$. We now apply a perturbation

$$\hat{H}_p = \frac{1}{2}m\omega'^2\hat{x}^2$$

where $\omega' \ll \omega$. Calculate the energy shifts through second order and compare with the exact eigenvalues. You will have to recall these formulas [**25**]

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad (1)$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \quad (2)$$