(a)

Q1. $E_x(t) = E_0 \sin kz \sin \omega t = \left(\frac{2\omega^2}{\varepsilon_0 V}\right)^{1/2} \sin kz \ (\hat{q}(t))$

[5 marks]

$E_x = E_\omega \sin kz \ (\hat{a} + \hat{a}^\dagger)$

where $E_\omega = \left(\frac{\hbar\omega}{\varepsilon_0 V}\right)^{1/2}$ ; $\hat{q} = \left(\frac{\hbar}{2\omega}\right)^{1/2} (\hat{a} + \hat{a}^\dagger)$.

$\langle n \rangle \propto E_0 \sin kz \ \langle n | (\hat{a} + \hat{a}^\dagger) | n \rangle = 0$

Q2.(a) $\rightarrow$ $\hat{F} = (\hat{n}+1)^{-1/2} \hat{a} = (\hat{a}^\dagger \hat{a} + 1)^{-1/2} \hat{a}$

$= (\hat{a}\hat{a}^\dagger)^{-1/2} \hat{a}$

$[\hat{a}\hat{a}^\dagger = 1 + \hat{a}^\dagger \hat{a}$

$\rightarrow$ $\hat{F} |\phi\rangle = e^{i\phi} |\phi\rangle$ define a phase eigenstate.

$$|\phi\rangle = \sum_{n=0}^{\infty} e^{in\phi} |n\rangle$$

$$\hat{F} |\phi\rangle = (\hat{a}\hat{a}^\dagger)^{-1/2} \hat{a} \sum_n e^{in\phi} |n\rangle$$

$$= \sum_n e^{in\phi} (\hat{a}\hat{a}^\dagger)^{-1/2} \hat{a} |n\rangle$$

$$= \sum_n \sqrt{n}\, e^{in\phi} (\hat{a}\hat{a}^\dagger)^{-1/2} |n-1\rangle$$

$\rightarrow$ $\hat{F} |n\rangle = (\hat{a}\hat{a}^\dagger)^{-1/2} \hat{a} |n\rangle$

$$= (\hat{n}+1)^{-1/2} \hat{a} |n\rangle = \begin{cases} 0, & n = 0 \\ \sqrt{n}\, (\hat{n}+1)^{-1/2} |n-1\rangle, & n \neq 0 \end{cases}$$

$$\boxed{\hat{F} |n\rangle = \begin{cases} 0, & n = 0 \\ |n-1\rangle, & n \neq 0 \end{cases}}$$

Q2(b)

when $n \neq 0$

$$\hat{F}|n\rangle = (\hat{n}+1)^{-1/2}\hat{a}|n\rangle = \sqrt{n}(\hat{n}+1)^{-1/2}|n-1\rangle$$

$$= \sqrt{n}(n-\cancel{1}+\cancel{1})^{-1/2}|n-1\rangle$$

$$= |n-1\rangle .$$

$$\hat{F}|n\rangle = \cancel{|n-1\rangle\langle n|} \left(\sum_{m=0}^{\infty} |m\rangle\langle m+1|\right)|n\rangle$$

$$= \sum_{m=0}^{\infty} |m\rangle\langle m+1|n\rangle = \sum_{m=0}^{\infty} |m\rangle \delta_{m,n-1}$$

$$= |n-1\rangle .$$

$$\therefore \boxed{\hat{F} = \sum_{m=0}^{\infty} |m\rangle\langle m+1|}$$

Q 2.(c)

$$\hat{F} \sum_{n=0}^{\infty} e^{in\phi}|n\rangle$$

$$= \left(\sum_{m=0}^{\infty} |m\rangle\langle m+1|\right)\left(\sum_{n=0}^{\infty} e^{in\phi}|n\rangle\right)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |m\rangle\langle m+1|n\rangle e^{in\phi}$$

$$= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \delta_{m,n-1}\, e^{in\phi}\, |m\rangle$$

(n cannot be zero because m+1 has a minimum of 1).

$$= \sum_{n=1}^{\infty} e^{i\,n\,\phi}\, |n-1\rangle$$

$$= \sum_{n=1}^{\infty} e^{in\phi}|n-1\rangle$$

$n-1 = n'$

$n = n'+1$

$$= \sum_{n'=0}^{\infty} e^{i\phi} e^{in'\phi}|n'\rangle$$

$$= e^{i\phi}\sum_{n'=0}^{\infty} e^{in'\phi}|n'\rangle$$

$$= e^{i\phi}\left(\sum_{n=0}^{\infty} e^{in\phi}|n\rangle\right)$$

Q2(d)

$$|\psi\rangle = \sum_{n=0}^{\infty} C_n |n\rangle \qquad \sum_n |C_n|^2 = 1$$

$$\langle \phi | \psi \rangle = \left( \sum_{n=0}^{\infty} e^{-in\phi} \langle n| \right) \left( \sum_{m=0}^{\infty} c_m |m\rangle \right)$$

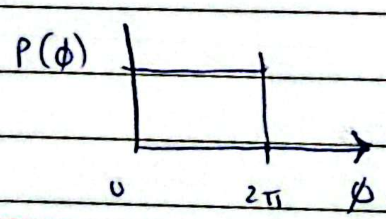
$$= \sum_{n=0}^{\infty} e^{-in\phi} c_n$$

$$\rightarrow P(\phi) = \frac{1}{2\pi} |\langle \phi | \psi \rangle|^2 = \frac{1}{2\pi} \left| \sum e^{-in\phi} c_n \right|^2$$

$C_n = 1$ for $n$.

$$P(\phi) = \frac{1}{2\pi} |e^{-in\phi}|^2 = \frac{1}{2\pi}.$$

$\rightarrow$



$$\langle \phi \rangle = \pi$$

$$\cancel{\langle \phi^2 \rangle = \frac{1}{2\pi} \cdot \frac{1}{2\pi} \int_0^{2\pi} d\phi\, \phi^2 = \frac{1}{4\pi^2} \cdot \frac{1}{3} (2\pi)^3 = \frac{8\pi^3}{3 \times 4\pi^2} = 2\pi/3.}$$

Q2(e)

$$\langle \phi^2 \rangle = \int P(\phi) \phi^2 d\phi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \phi^2 d\phi$$

$$= \frac{1}{2\pi} \cdot \frac{1}{3} 8\pi^3 = \frac{4\pi^2}{3}$$

$$(\Delta\phi)^2 = \frac{4\pi^2}{3} - \pi^2 = \frac{\pi^2}{3} \Rightarrow \boxed{\Delta\phi = \frac{\pi}{\sqrt{3}}}$$

~~Q4.~~ Q3. [10 marks]

$$\cancel{\frac{d}{dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = -\vec{\Omega} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} - \gamma \begin{pmatrix} u \\ v \\ w \end{pmatrix}} \ell(w+1)$$

$$\delta\omega = \Omega_R, \quad \Omega = \sqrt{\Omega_R^2 + \delta\omega^2} = \sqrt{2}\,\Omega_R.$$

$$U = \exp\left(-i\sqrt{2}\,\Omega_R \frac{1}{2}\left(\cancel{\Omega_R} \frac{1}{\sqrt{2}}\hat{\sigma}_x + \frac{1}{\sqrt{2}}\hat{\sigma}_z\right)\right)$$

$$\cancel{\hat{U} = \exp\left(-i\frac{\Omega_R}{2}(\hat{\sigma}_x + \hat{\sigma}_z)t\right)}$$

$$\cancel{\hat{U}|a\rangle = \exp\left(-i\frac{\Omega_R}{2}(\hat{\sigma}_x + \hat{\sigma}_z)t\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$\cancel{\hat{U} = \exp} \quad (\hat{\sigma}_x + \hat{\sigma}_z) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$(\hat{\sigma}_x + \hat{\sigma}_z)^2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\hat{U} = \exp\left(-i\frac{\Omega_R\sqrt{2}}{2}\left(\frac{\hat{\sigma}_x + \hat{\sigma}_z}{\sqrt{2}}\right)t\right) = \cos\left(\frac{\Omega_R t}{\sqrt{2}}\right)\cancel{\left(\frac{\hat{\sigma}_x + \hat{\sigma}_z}{\sqrt{2}}\right)}\hat{1}$$

$$-i\left[\sin\left(\frac{\Omega_R t}{\sqrt{2}}\right)\right]\left(\frac{\hat{\sigma}_x + \hat{\sigma}_z}{\sqrt{2}}\right)$$

$$|f\rangle = \hat{U}|a\rangle$$

$$= \cos\left(\frac{\Omega_R t}{\sqrt{2}}\right)\hat{1}\,|a\rangle - i\sin\left(\frac{\Omega_R t}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\left(\hat{\sigma}_x + \hat{\sigma}_z\right)|a\rangle$$

$$= \cos\left(\frac{\Omega_R t}{\sqrt{2}}\right)|a\rangle - i\sin\left(\frac{\Omega_R t}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\left(|b\rangle + |a\rangle\right)$$

$$= \left(\cos\left(\frac{\Omega_R t}{\sqrt{2}}\right) - i\sin\left(\frac{\Omega_R t}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\right)|a\rangle - i\sin\left(\frac{\Omega_R t}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}|b\rangle.$$

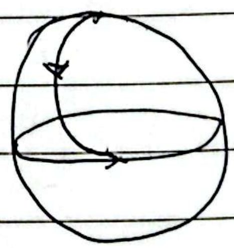
at $\Omega_R t = \left(\frac{\pi}{2}\sqrt{2}\right)$ → will take longer than $\pi/2$.

$$|f\rangle = \left(0 - \frac{i}{\sqrt{2}}\right)|a\rangle - \frac{i}{\sqrt{2}}|b\rangle = -\frac{i}{\sqrt{2}}\left(|a\rangle + |b\rangle\right)$$

→ Show on Bloch sphere.

As $\hat{\sigma}_x|a\rangle = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = |b\rangle$

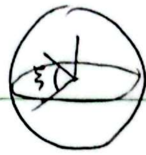
$\hat{\sigma}_z|a\rangle = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = |a\rangle$



• If $\delta\omega = 2\Omega_R$,

$$\Omega = \sqrt{\Omega_R^2 + 4\Omega_R^2} = \Omega_R\sqrt{5}$$

$$\frac{\Omega_R\sqrt{5}}{2}t\Big|_{max} = \pi/2 \Rightarrow \boxed{t = \frac{\pi}{\Omega_R\sqrt{5}}} \Rightarrow \Omega_R t = \frac{\pi}{2}\cdot\frac{2}{\sqrt{5}}$$




$$U = \exp\left(-i \frac{\Omega_R \sqrt{5} t}{2} \left(\cos \xi \, \hat{\sigma}_x + \sin \xi \, \hat{\sigma}_z\right)\right)$$

where $\cos \xi = \frac{\Omega_R}{\Omega}$, $\sin \xi = \frac{\delta\omega}{\Omega}$.

$$\Omega = \Omega_R \sqrt{5}$$

$$\cos \xi = \frac{1}{\sqrt{5}}, \quad \sin \xi = \cancel{5\Omega_R} \; \frac{2\Omega_R}{\sqrt{5}\,\Omega_R} = \frac{2}{\sqrt{5}}.$$

$$U = \exp\left(-i \frac{\Omega_R \sqrt{5} t}{2} \left(\frac{\hat{\sigma}_x}{\sqrt{5}} + \frac{2}{\sqrt{5}} \hat{\sigma}_z\right)\right)$$

For $t = \frac{\pi \sqrt{2}}{2 \Omega_R}$

$$U = \exp\left(-i \frac{\cancel{\Omega_R} \sqrt{5}}{2} \frac{\pi \sqrt{2}}{2 \cancel{\Omega_R}} \left(\frac{\hat{\sigma}_x}{\sqrt{5}} + \frac{2}{\sqrt{5}} \hat{\sigma}_z\right)\right)$$

$$= \exp\left(- \frac{i \pi \sqrt{10}}{4\sqrt{5}} \left(\hat{\sigma}_x + 2\hat{\sigma}_z\right)\right)$$

$$= \exp\left(- \frac{i \pi \sqrt{2}}{4} \left(\hat{\sigma}_x + 2\hat{\sigma}_z\right)\right).$$

Q5

$$|\psi\rangle = \frac{|\alpha\rangle + |\alpha e^{i2\pi/3}\rangle + |\alpha e^{-i2\pi/3}\rangle}{\sqrt{3}}$$

$\langle n|\psi\rangle$

$\therefore$

$$\langle n|\alpha\rangle = \langle n|\left(e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}\right)|0\rangle$$

$$= e^{-|\alpha|^2/2}\langle n|e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}|0\rangle.$$

$$\langle n|\alpha\rangle = \langle n|\sum_{m=0}^{\infty} e^{-|\alpha|^2/2}\frac{(\alpha)^m}{\sqrt{m!}}|m\rangle$$

$$= e^{-|\alpha|^2/2}\langle n|\sum_{m=0}^{\infty}\frac{(\alpha)^m}{\sqrt{m!}}|m\rangle$$

$$= e^{-|\alpha|^2/2}\frac{(\alpha^n)}{\sqrt{n!}}$$

$$\langle n|e^{i2\pi/3}\alpha\rangle = e^{-|\alpha|^2/2}\frac{\alpha^n e^{i2\pi n/3}}{\sqrt{n!}}$$

$$\therefore \langle n|\psi\rangle = \frac{e^{-|\alpha|^2/2}}{\sqrt{3}\sqrt{n!}}\alpha^n\left(1 + e^{i2\pi n/3} + e^{-i2\pi n/3}\right)$$

$$= \frac{e^{-|\alpha|^2/2}\alpha^n\left(1 + 2\cos\left(2\pi n/3\right)\right)}{\sqrt{n!}\sqrt{3}}$$

$$|\langle n|\psi\rangle|^2 = \frac{e^{-|\alpha|^2}}{3\,n!}\,|\alpha|^2\left|\left(1+2\cos\left(2n\pi/3\right)\right)\right|^2 .$$

$\cos 120^\circ =$

$= \cos(180-60)$

$= -\cos 60$

$= -1/2$

$\cos(180+60^\circ)$

$=$

| $n$ | 0 | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|---|
| $\cos\left(\frac{2n\pi}{3}\right)$ | 1 | $-1/2$ | $-1/2$ | 1 | $-1/2$ | |
| $1+2\cos\left(\frac{2n\pi}{3}\right)$ | 3 | 0 | 0 | 3 | 0 | ... |
| $\left\lvert 1+2\cos\left(\frac{2n\pi}{3}\right)\right\rvert^2$ | 9 | 0 | 0 | 9 | 0 | |

$$|\langle n|\psi\rangle|^2 = \frac{e^{-|\alpha|^2}}{3}\,|\alpha|^2 \sum_{n=0}^{\infty} \frac{9}{(3n)!}$$

$$= 3e^{-|\alpha|^2}\,|\alpha|^2 \sum_{n=0}^{\infty} \frac{1}{(3n)!} .$$