Quantum Optics: HW5 by Muhammad Sabieh Anwar

Please submit before 10 am, Wednesday, 30 April 2025; only by the channel communicated.

These homework problems use a number of identities that can be derived using the Baker Campbell lemma. You are welcome to utilize these identities from one question to the next. The squeeze operator is defined as $\hat{S}(\xi) = \exp\left(\xi^* \hat{a}^2 - \xi \hat{a^{\dagger 2}}\right)$ and the displacement operator is $\hat{D}(\alpha) = \exp\left(\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right)$.

1. Prove that $\hat{S}(\xi)$ is unitary.

2. Do $\hat{S}(\xi)$ and $\hat{D}(\alpha)$ commute? Don't believe what I told you in class.

3. Show that the squeezed vacuum state in the field quadrature picture "breathes"! This means that I expect that you determine the uncertainties $(\Delta X_1)^2(t)$ and $(\Delta X_2)^2(t)$ for the squeezed vacuum $|\xi\rangle = \hat{S}(\xi)|0\rangle$. Show the complete working and employ the time-dependent form of the field quadratures,

$$\hat{X}_1(t) = \frac{1}{2}(\hat{a}(t) + \hat{a}^\dagger(t)) \tag{1}$$

$$\hat{X}_2(t) = \frac{1}{2i}(\hat{a}(t) - \hat{a}^\dagger(t)). \tag{2}$$

4. This question deals with a displaced squeezed vacuum state $|\chi\rangle = \hat{D}(\alpha)\,\hat{S}(\xi)\,|0\rangle$. Note that $\xi = re^{i\theta}$, $(r = |\xi|)$ and $\alpha = |\alpha|e^{i\psi}$.

   (a) Find expressions for

$$\hat{D}(\alpha)\,\hat{a}\,\hat{D}^\dagger(\alpha) \tag{3}$$

$$\hat{D}(\alpha)\,\hat{a}^\dagger\,\hat{D}^\dagger(\alpha) \tag{4}$$

$$\hat{S}(\xi)\,\hat{a}\,\hat{S}^{\dagger}(\xi) \tag{5}$$

$$\hat{S}(\xi)\,\hat{a}^{\dagger}\,\hat{S}^{\dagger}(\xi). \tag{6}$$

(b) We want to see what the state $|\chi\rangle$ looks like when expressed in terms of the number states. This allows us to calculate the photon distribution. Assume that:

$$|\chi\rangle = \sum_n C_n\,|n\rangle, \tag{7}$$

and utilizing the annihilation of vacuum $\hat{a}|0\rangle = 0$, derive the following recursion relation for the coefficients,

$$\mu\sqrt{n+1}\,C_{n+1} - \gamma\,C_n + \nu\sqrt{n}\,C_{n-1} = 0, \tag{8}$$

where $\mu = \cosh r$, $\nu = e^{i\theta}\sinh r$ and $\gamma = \alpha\mu + \alpha^{*}\nu$.

(c) Assume an ansatz for the coefficient

$$C_n = \mathcal{N}(\frac{\nu}{2\mu})^{n/2}\,\frac{1}{\sqrt{\mu}}\,f_n(x), \tag{9}$$

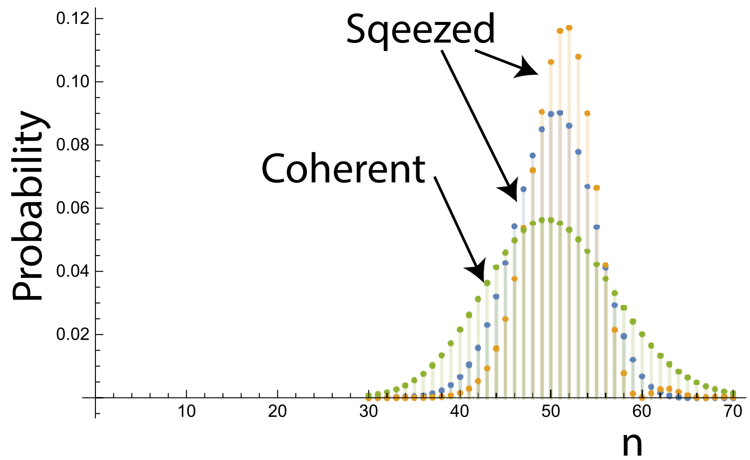
and show that the recursive relation in Eq. (8) can be written in terms of the unknwon function f_n as:

$$\sqrt{n+1}f_{n+1}(x) - 2\gamma\,(e^{i\theta}\,\sinh 2r)^{-1/2}\,f_n(x) + 2\sqrt{n}\,f_{n-1}(x). \tag{10}$$

(d) Show that Eq. (10) satisfies the recursion relationship for the well-known Hermite polynomial $H_n(x)$:

$$H_{n+1}(x) - 2\,x\,H_n(x) + 2\,n\,H_{n-1}(x) = 0 \tag{11}$$

provided $f_n(x) = H_n(x)/\sqrt{n!}$ and $x = \gamma\,(e^{i\theta}\sinh 2r)^{-1/2}$.



(e) Hence write the state $|\chi\rangle$ in Eq. (7) in terms of the Hermite polynomial with the yet-to-be determined normalization constant $\mathcal{N}$. We now wish to determine $\mathcal{N}$. For this purpose compute $C_0 = \langle 0|\chi\rangle = \langle 0|\hat{D}(\alpha)\hat{S}(\xi)|0\rangle$ and hence calculate $\mathcal{N}$.

(f) Determine the probability distribution of photon states for the state $|\chi\rangle$. Plot. For plotting you may like to use the values, $\theta = \pi/3$, $r = 1/2$, $|\alpha| = \sqrt{50}$. Try $\psi = \pi/3$. Also vary r from 0.5 to 0.9. What about changing ψ to $2\pi/3$? Are the photon statistics sub or super-Poissonian and how do they compare to the Poissonian statistics observed for coherent light which is *not* squeezed? Some typical results I obtained are shown in the accompanying diagram.