**Quantum Optics: Final Take-Home Exam** by

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*Please submit before 10 am, Monday, 19 May 2025; only by the channel communicated.*

Attempt all questions.

1. *Second order correlation function $g^2(\tau)$.* Given that the fluctuating cycle-averaged intensity at time $t$ can be described by:

$$\overline{I}(t) = \langle \overline{I}(t) \rangle + \Delta \overline{I}(t), \tag{1}$$

write an expression for $g^2(\tau)$ and show that this correlation function is always equal to or greater than 1. You are allowed to start only by using Eq. (1).

2. *An extension to the Jaynes-Cummings model..* Consider an atom-photon interaction, described in the interaction frame by,

$$\tilde{\hat{H}} = \hbar\lambda(\hat{a}^4\hat{\sigma}^+ + \hat{a}^{\dagger 4}\hat{\sigma}^-). \tag{2}$$

   (a) Given the manifold of bare states $|n, e\rangle$ and $|n, g\rangle$, identify the dressed states of the system.

   (b) What are the energies of the dressed states?

   (c) Suppose the atom-photon system is initialized in the quantum state $|n + 4, g\rangle$. What is the state after time $t$?

   (d) What is the (on-resonance) Rabi frequency of the atom-photon system?

3. *Beamsplitter and interferometer.* Consider the accompanying diagram. The three modes involved in this interferometer are labeled as $a$, $b$ and $c$ as shown. The action of the beamsplitter was described in class and we stick to that definition. The "phase" element introduces an interaction

$$\hat{U}_{\text{phase}} = \exp\left(-i\phi\hat{a}^{\dagger}\hat{a}\right) \tag{3}$$

and the nonlinear optical element denoted as "Kerr" introduces an interaction,

$$\hat{U}_{\text{Kerr}} = \exp\left(-iKt\,\hat{b}^{\dagger}\hat{b}\,\hat{c}^{\dagger}\hat{c}\right), \tag{4}$$

where $K$ is a constant and $t$ is the time.

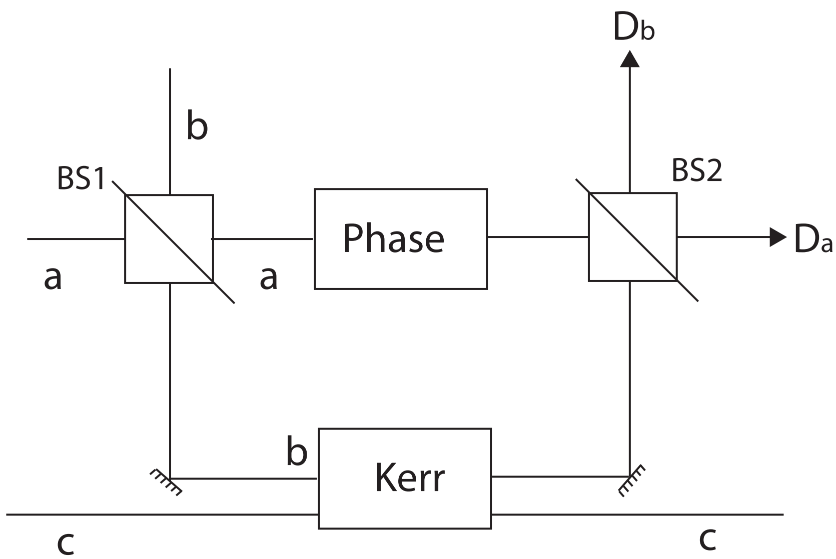


Figure 1: Diagram for Question 3.

(a) The input to this interferometer is $|1\rangle_a|0\rangle_b|\alpha\rangle_c$ where $|\alpha\rangle$ is a coherent state. Show the progression of the quantum state through the interferometer, and note down the state just before the second beamsplitter.

(b) What is the state before the beamsplitter when $\phi = \pi$ and when $Kt = \pi$ simultaneously?

(c) With these settings of the phase and duration of interaction, what is the quantum state after the second beamsplitter? Is this an entangled state?

(d) If the detector $D_a$ clicks, what is the state of the mode $c$? What if the detector $D_b$ clicks. This is a counterintuitive example of a projective measurement producing a superposition!

4. *Generation of squeezed light.* Consider the accompanying diagram. A single mode labeled as $a$ comes into a nonlinear material and a single mode $b$ emerges, or the other way round. The two modes have frequencies $\omega_a$ and $\omega_b$ respectively. The Hamiltonian that describes this interaction is given by,

$$\hat{H} = \hbar\omega_a \hat{a}^\dagger \hat{a} + \hbar\omega_b \hat{b}^\dagger \hat{b} + i\hbar g(\hat{a}\hat{b}^{\dagger 2} - \hat{a}^\dagger \hat{b}^2), \tag{5}$$

where $g$ is a coupling parameter connecting the two modes.

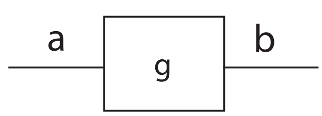


Figure 2: Diagram for Question 4.

(a) Suppose the input is a strong amplitude coherent state $|a\rangle$. In this condition, $\hat{a}$ can be replaced by $\alpha e^{-i\omega_a t}$ and likewise for the adjoint. This is called the parametric approximation. What does the Hamiltonian (5) look like when this happens?

(b) Write the Hamiltonian in the interaction frame.

(c) What happens when $\omega_a = 2\omega_b$?

(d) Does the action of the nonlinear material resemble the squeeze operator?

5. *Photon number distribution.* How does a coherent state $|\alpha\rangle$ evolve under the action of a Hamiltonian

$$\hat{H} = \hbar K \hat{a}^{\dagger 2} \hat{a}^2? \tag{6}$$

Show that the photon distribution remains Poissonian for the evolved state. The complete working is requested. The constant $K$ shows the amount of nonlinearity.