HW5

Q1. Is $S(\xi) = e^{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2})}$ unitary?

$$S(\xi) = e^{\frac{1}{2}\xi^* \hat{a}^2} e^{-\frac{1}{2}\xi \hat{a}^{\dagger 2}} e^{+\frac{|\xi|^2}{4}[\hat{a}^2, \hat{a}^{\dagger 2}]/2}$$ BCH.

$$\because e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-[\hat{A},\hat{B}]/2}$$

$$S(\xi) = e^{\frac{1}{2}\xi^* \hat{a}^2} e^{-\frac{1}{2}\xi \hat{a}^{\dagger 2}} e^{+\frac{|\xi|^2}{8}[\hat{a}^2, \hat{a}^{\dagger 2}]}$$

$$[\hat{a}^2, \hat{a}^{\dagger 2}] = \hat{a}\hat{a}\hat{a}^\dagger\hat{a}^\dagger - \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}$$

$$= \hat{a}(1+\hat{a}^\dagger\hat{a})\hat{a}^\dagger - \hat{a}^\dagger(\hat{a}\hat{a}^\dagger - 1)\hat{a}$$

$$= \underline{\hat{a}\hat{a}^\dagger} + a a^\dagger a \hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{a}^\dagger a - \underline{\hat{a}^\dagger\hat{a}}$$

$$= 1 - (\hat{a}^\dagger\hat{a})(\hat{a}^\dagger\hat{a}) + (1+\hat{a}^\dagger\hat{a})(1+\hat{a}^\dagger\hat{a})$$

$$= 1 - \cancel{(a^\dagger a)^2} + 1 + 2\hat{a}^\dagger\hat{a} + \cancel{(a^\dagger a)^2}$$

$$= 2(1+a^\dagger\hat{a}) = 2\hat{a}\hat{a}^\dagger$$

$$\boxed{[\hat{a}^2, \hat{a}^{\dagger 2}] = 2\hat{a}\hat{a}^\dagger}$$

$$\therefore S(\xi) = e^{\frac{1}{2}\xi^* \hat{a}^2} e^{-\frac{1}{2}\xi \hat{a}^{\dagger 2}} e^{+\frac{|\xi|^2}{4}(\hat{a}\hat{a}^\dagger)} \quad \text{— (1)}$$

Now $S^{-1} = S(-\xi) = e^{\frac{1}{2}(\xi \hat{a}^\dagger - \xi^* \hat{a}^2)}$.

Now $S^\dagger(\xi) = e^{\frac{1}{2}(\xi \hat{a}^{\dagger 2} - \xi^* \hat{a}^2)}$

$$= \cancel{e^{\frac{1}{2}\xi \hat{a}^{\dagger 2}}} \; e^{\frac{1}{2}(-\xi^* \hat{a}^2 + \xi \hat{a}^{\dagger 2})}$$

$$= e^{-\frac{1}{2}\xi^* \hat{a}^2} e^{+\frac{\xi}{2}\hat{a}^{\dagger 2}} e^{+\frac{1}{4}|\xi|^2 [\hat{a}^2, \hat{a}^{\dagger 2}]/2}$$

From ①

$$S(-\xi) = e^{-\frac{1}{2}\xi^* \hat{a}^2} e^{+\frac{1}{2}\xi \hat{a}^{\dagger 2}} e^{\frac{|\xi|^2}{4}(a\hat{a}^\dagger)}$$

$$\downarrow$$

$$e^{\frac{1}{4}|\xi|^2 [\hat{a}^2, \hat{a}^{\dagger 2}]/2}$$

Hence $\hat{S}^\dagger(\xi) = \hat{S}(-\xi)$

$\hat{S}$ is unitary.

Q2. $\boxed{\text{Do } D \text{ and } S \text{ commute?}}$

$$\hat{D}(\alpha) S(\xi) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} e^{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2})}$$

reverse the order

$$= e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a} + \frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2}) + \frac{1}{2}\left[\alpha \hat{a}^\dagger - \alpha^* \hat{a}, \frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2})\right]}$$

$$\hat{S}(\xi)\hat{D}(\alpha) = \left(e^{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2})}\right)\left(e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}\right) e^{+\left[\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2}), \alpha \hat{a}^\dagger - \alpha^* \hat{a}\right]}$$

F

Find $= \left[\alpha \hat{a}^\dagger - \alpha^* \hat{a}, \xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2}\right] = \alpha \left[\hat{a}^\dagger, \xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2}\right]$
$- \alpha^* \left[\hat{a}, \xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2}\right]$

$$\left[\hat{a}^\dagger, \hat{a}^2\right] = \hat{a}^\dagger \hat{a}\hat{a} - \hat{a}\hat{a}\hat{a}^\dagger$$
$$= \hat{a}^\dagger \hat{a}\hat{a} - a(1 + \hat{a}^\dagger \hat{a}) = \hat{a}^\dagger \hat{a}\hat{a} - \hat{a} - \hat{a}\hat{a}^\dagger \hat{a}$$
$$= (\hat{a}^\dagger a - \hat{a}\hat{a}^\dagger)\hat{a} - \hat{a} = \underline{-2\hat{a}}$$

and

$$\left[\hat{a}, \hat{a}^{\dagger 2}\right] = \hat{a}\hat{a}^\dagger \hat{a}^\dagger - \hat{a}^\dagger \hat{a}^\dagger \hat{a}$$
$$= \hat{a}\hat{a}^\dagger \hat{a}^\dagger - \hat{a}^\dagger(\hat{a}\hat{a}^\dagger - 1) = \hat{a}\hat{a}^\dagger \hat{a}^\dagger - \hat{a}^\dagger \hat{a}\hat{a}^\dagger + \hat{a}^\dagger = \underline{+2\hat{a}^\dagger}$$

$$\therefore \hat{F} = \alpha \xi^* [\hat{a}^\dagger, \hat{a}^2] + \alpha^* \xi [\hat{a}, \hat{a}^{\dagger 2}]$$

$$= \alpha \xi^* (-2\hat{a}) + \alpha^* \xi \, 2\hat{a}^\dagger$$

$$= 2\alpha^* \xi \, \hat{a}^\dagger - 2\alpha \xi^* \hat{a} \neq 0$$

They don't commute. Order matters.

Q3.

Show that the squeezed vacuum state $|\xi\rangle = \hat{S}(\xi)|0\rangle$ breathes! This means that we want to see how $(\Delta \hat{X}_1)^2(t)$ and $(\Delta \hat{X}_2)^2(t)$ oscillate in time. Show the complete working. Use the time dependent forms of the quadratures;

$$\hat{X}_1(t) = \frac{1}{2}(\hat{a}(t) + \hat{a}^\dagger(t)), \text{ and}$$

$$\hat{X}_2(t) = \frac{1}{2i}(\hat{a}(t) - \hat{a}^\dagger(t)).$$

$$|\xi\rangle = \hat{S}(\xi)|0\rangle$$

$$\langle X_1\rangle^{(t)} = \frac{1}{2}\langle \hat{a}(t) + \hat{a}^\dagger(t)\rangle_\xi$$

$$= \langle 0|\hat{S}^\dagger(\xi)\left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{+i\omega t}\right)\hat{S}(\xi)|0\rangle$$

We need $\hat{S}^\dagger \hat{a} \hat{S}$ and $\hat{S}^\dagger \hat{a}^\dagger \hat{S}$.

$$\hat{S}^\dagger = e^{-\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2})}$$

Let $-\frac{1}{2}\left(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2}\right) = A$.

Then $e^{A}\,\hat{a}\,e^{-A} = \hat{a} + [\hat{A},\hat{a}] + \frac{1}{2}[\hat{A},[\hat{A},\hat{a}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{a}]]] + \frac{1}{4!}[\hat{A},[\hat{A},[\hat{A},[\hat{A},\hat{a}]]]] + \dots$

Now $[\hat{A},\hat{a}] = \left[-\frac{1}{2}\xi^* \hat{a}^2 + \frac{1}{2}\xi \hat{a}^{\dagger 2}, \hat{a}\right]$

$= \frac{1}{2}\xi\left[\hat{a}^{\dagger 2},\hat{a}\right] = -\xi \hat{a}^{\dagger}$.

$[A,[\hat{A},\hat{a}]] = \left[-\frac{1}{2}\xi^* \hat{a}^2 + \frac{\xi}{2}\hat{a}^{\dagger 2}, -\xi \hat{a}^{\dagger}\right] = +\frac{1}{2}|\xi|^2\left[\hat{a}^2,\hat{a}^{\dagger}\right]$

$= +|\xi|^2 \hat{a}$.

$[\hat{A},[\hat{A},[\hat{A},\hat{a}]]] = \left[-\frac{1}{2}\xi^* \hat{a}^2 + \frac{\xi}{2}\hat{a}^{\dagger 2}, +|\xi|^2 \hat{a}\right]$

$= +\frac{1}{2}|\xi|^2 \xi\left[\hat{a}^{\dagger 2},\hat{a}\right] = |\xi|^2 \xi \hat{a}^{\dagger}$.

$[\hat{A},[\hat{A},[\hat{A},[\hat{A},\hat{a}]]]] = \left[-\frac{1}{2}\xi^* \hat{a}^2 + \frac{\xi}{2}\hat{a}^{\dagger 2}, -|\xi|^2 \xi \hat{a}^{\dagger}\right]$

$= +\frac{1}{2}|\xi|^4\left[\hat{a}^2,\hat{a}^{\dagger}\right] = +|\xi|^4 \hat{a}$.

$\therefore\ S^{\dagger}\hat{a}S = \hat{a} - \xi \hat{a}^{\dagger} + \frac{1}{2}|\xi|^2 \hat{a} - \frac{1}{3!}|\xi|^2 \xi \hat{a}^{\dagger} + \frac{1}{4!}|\xi|^4 \hat{a} + \dots$

Now $1 + \frac{1}{2}|\xi|^2 + \frac{1}{4!}|\xi|^4 \dots = \cosh x$

$\cosh x = \frac{e^{x} + e^{-x}}{2} = \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right)\right]$

$= \frac{1}{2}\left(1 + x^2/2! + x^4/4! + \dots\right)$

Also $\frac{e^{r}-e^{-r}}{2} = \frac{1}{2}\left(1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \cdots\right)$

$$- \frac{1}{2}\left(1 - r + \frac{r^2}{2!} - \frac{r^3}{3!} + \frac{r^4}{4!} - \cdots\right)$$

$$\sinh r = r + \frac{r^3}{3!} + \frac{r^5}{5!} + \cdots$$

$$- \xi \sinh r \; a^{\dagger}$$

$$S^{\dagger} a S = \hat{a}\left(1 + \frac{1}{2}|\xi|^2 + \frac{1}{4}|\xi|^4 + \cdots\right)$$

$$- \hat{a}^{\dagger}\left(\xi + \frac{\xi|\xi|^2}{3!} + \frac{\xi|\xi|^4}{5!} + \cdots\right)$$

$$= \hat{a} \cosh r - \hat{a}^{\dagger}\left(r e^{i\theta} + \frac{r^3 e^{i\theta}}{3!} + \frac{r^5 e^{i\theta}}{5!} + \cdots\right)$$

$$S^{\dagger} \hat{a} S = \hat{a} \cosh r - e^{i\theta} \hat{a}^{\dagger} \sinh r$$

i.e $\boxed{S^{\dagger}(\xi)\hat{a}\, S(\xi) = \hat{a} \cosh \hat{r} - \hat{a}^{\dagger} e^{i\theta} \sinh r}$

Similarly, for $S^{\dagger} \hat{a}^{\dagger} S$ we ~~put~~ replace $\hat{a}$ by $\hat{a}^{\dagger}$ in the previous calculation.

$$[\hat{A}, \hat{a}^{\dagger}] = -\frac{1}{2}\xi^{*}[\hat{a}^2, \hat{a}^{\dagger}] = -\xi^{*}\hat{a}$$

$$[\hat{A}, [\hat{A}, \hat{a}^{\dagger}]] = \left[-\frac{1}{2}\xi^{*}\hat{a}^2 + \frac{1}{2}\xi \hat{a}^{\dagger 2}, -\xi^{*}\hat{a}\right] = -\frac{1}{2}|\xi|^2 [\hat{a}^{\dagger 2}, \hat{a}]$$

$$= +|\xi|^2 \hat{a}^{\dagger}$$

$$[\hat{A},[\hat{A},[\hat{A},\hat{a}^\dagger]]] = \left[-\tfrac{1}{2}\xi^*\hat{a}^2 + \tfrac{1}{2}\xi\hat{a}^{\dagger 2},\ +|\xi|^2\hat{a}^\dagger\right]$$

$$= -\tfrac{1}{2}\xi^*|\xi|^2\, 2\hat{a} = -\xi^*|\xi|^2\hat{a}$$

$$[\hat{A},[\hat{A},[\hat{A},[\hat{A},\hat{a}^\dagger]]]] = \left[-\tfrac{1}{2}\xi^*\hat{a}^2 + \tfrac{1}{2}\xi\hat{a}^{\dagger 2},\ -\xi^*|\xi|^2\hat{a}\right]$$

$$= -\tfrac{1}{2}|\xi|^4\left[\hat{a}^{\dagger 2},\hat{a}\right] = +|\xi|^4\hat{a}^\dagger$$

$$S^\dagger\hat{a}^\dagger S = a^\dagger - \xi^*\hat{a} + \frac{|\xi|^2}{2}\hat{a}^\dagger - \xi^*\frac{|\xi|^2}{3!}\hat{a} + \frac{|\xi|^4}{4!}\hat{a}^\dagger - \cdots$$

$$= \hat{a}^\dagger\left(1 + \frac{|\xi|^2}{2!} + \frac{|\xi|^4}{4!} - \cdots\right)$$

$$- \hat{a}\left(\xi^* + \xi^*\frac{|\xi|^2}{3!} + \xi^*\frac{|\xi|^3}{5!} + \cdots\right)$$

$$\boxed{\hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{S}(\xi) = \hat{a}^\dagger\cosh r - \hat{a}\,e^{-i\theta}\sinh r}$$

$$\therefore \langle\hat{X}_1(t)\rangle = e^{-i\omega t}\langle 0|(\hat{a}\cosh r - \hat{a}^\dagger e^{i\theta}\sinh r)|0\rangle$$

$$+ e^{+i\omega t}\langle 0|(\hat{a}^\dagger\cosh r - \hat{a}e^{-i\theta}\sinh r)|0\rangle$$

$$\langle\hat{X}_1\rangle(t) = \cancel{e^{-i\omega t}e^{i\theta}\sinh r + e^{i\omega t}\cosh r} = 0$$

Likewise $\langle\hat{X}_2\rangle(t) = 0$

$$\langle\hat{X}_1^2\rangle(t) = \frac{1}{4}\langle\xi|(\hat{a}^2 e^{-i2\omega t} + \hat{a}^{\dagger 2}e^{+i2\omega t} + 1 + 2\hat{a}^\dagger\hat{a})|\xi\rangle$$

Quick way to compute $\langle 0| S^\dagger \hat{a}^\dagger \hat{a} S$

$$= \left(\hat{S}^\dagger \hat{a}^\dagger \hat{S}\right)\left(\hat{S}^\dagger \hat{a} \hat{S}\right)$$

$$= \left(\hat{a}^\dagger \cosh r - \hat{a}\, e^{-i\theta} \sinh r\right)\left(\hat{a} \cosh r - \hat{a}^\dagger e^{i\theta} \sinh r\right)$$

$$= \hat{a}^\dagger \hat{a} \cosh^2 r - \hat{a}^{\dagger 2} e^{i\theta} \sinh r \cosh r - \hat{a}^2 e^{-i\theta} \sinh r \cosh r + \left(1 + \hat{a}^\dagger \hat{a}\right) \sinh^2 r$$

$$= \sinh^2 r + \underline{\hat{a}^\dagger \hat{a}} \left(\cosh^2 r + \sinh^2 r\right) - \sinh r \cosh r \left(e^{i\theta} \hat{a}^{\dagger 2} + e^{-i\theta} \hat{a}^2\right)$$

$$\boxed{\langle 0| \left(\hat{S}^\dagger \hat{a}^\dagger S\right)\left(S^\dagger \hat{a} S\right)|0\rangle = \sinh^2 r}$$

$$\cancel{\langle \Delta x_1^2 \rangle (t) = \langle \hat{x}_1^2 \rangle (t) = \frac{1}{4}\left(1 + 2\sinh^2 r\right)}$$

similarly

$$\left(\hat{S}^\dagger \hat{a}^{\dagger 2} \hat{S}\right) = \left(\hat{S}^\dagger \hat{a} S\right)\left(S^\dagger \hat{a} S\right)$$

$$= \left(a \cosh r - \hat{a}^\dagger e^{i\theta} \sinh r\right)\left(a \cosh r - \hat{a}^\dagger e^{i\theta} \sinh r\right)$$

$$= \hat{a}^2 \cosh^2 r - \hat{a}\hat{a}^\dagger e^{i\theta} \cosh r \sinh r$$

$$- \hat{a}^\dagger \hat{a}\, e^{i\theta} \cosh r \sinh r$$

$$+ \hat{a}^{\dagger 2} e^{2i\theta} \sinh^2 r$$

$$(\hat{S}^\dagger \hat{a}^\dagger \hat{S}) = (S^\dagger \hat{a}^\dagger S)(\hat{S}^\dagger \hat{a}^\dagger \hat{S})$$

$$= (\hat{a}^\dagger \cosh r - \hat{a}\, e^{-i\theta} \sinh r)(\hat{a}^\dagger \cosh r - \hat{a}\, e^{-i\theta} \sinh r)$$

$$= \hat{a}^{\dagger 2} \cosh^2 r - \hat{a}^\dagger \hat{a}\, e^{-i\theta} \cosh r \sinh r - \hat{a}\hat{a}^\dagger e^{-i\theta} \cosh r \sinh r$$

$$+ \hat{a}^2 e^{-2i\theta} \sinh^2 r.$$

$$\therefore (\Delta X_1)^2(t) = \langle X_1^2 \rangle (t)$$

$$= \frac{1}{4}\Big[e^{-i2\omega t} e^{i\theta} \cosh r \sinh r \langle 0| -(\overbrace{\hat{a}\hat{a}^\dagger + \hat{a}^\dagger \hat{a}}^{1+2\hat{a}^\dagger \hat{a}})|0\rangle$$

$$+ e^{+i2\omega t} e^{-i\theta} \cosh r \sinh r \langle 0| -(\hat{a}^\dagger \hat{a} + \overbrace{\hat{a}\hat{a}^\dagger}^{1+2\hat{a}^\dagger a})|0\rangle$$

$$+ 1 + 2\sinh^2 r \Big]$$

$$= \frac{1}{4}\Big[-e^{-i2\omega t} e^{i\theta} \cosh r \sinh r - e^{+i2\omega t} e^{-i\theta} \cosh r \sinh r$$

$$+ 1 + 2\sinh^2 r \Big]$$

oscillatory functn.

At $\theta = 0$

$$(\Delta X_1)^2(t) = \frac{1}{4}\Big[-\cosh r \sinh r \cos(2\omega t) + 1 + 2\sinh^2 r \Big]$$

Shortcut for $\langle \hat{X}_2^2 \rangle (t)$.

(Note $\langle \hat{Y}_2 \rangle (t) = 0$).

$$\langle \hat{X}_2^2 \rangle (t) = -\frac{1}{4} \langle \xi | (\hat{a}(t) - \hat{a}^\dagger(t))^2 | \xi \rangle$$

$$= -\frac{1}{4} \langle \hat{a}^2 + \hat{a}^{\dagger 2} - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a} \rangle$$

$$= -\frac{1}{4} \langle \hat{a}^2 + \hat{a}^{\dagger 2} - (1 + 2\hat{a}^\dagger \hat{a}) \rangle$$

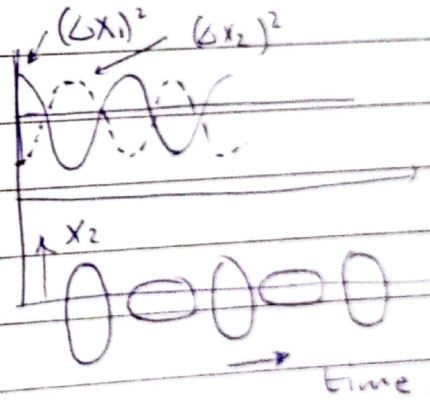
$$= -\frac{1}{4} \Big[-e^{-2i\omega t} e^{i\theta} \cosh r \sinh r - e^{+i2\omega t} e^{-i\theta} \cosh r \sinh r - 1 - 2\sinh^2 r \Big]$$

At $\theta = 0$

$$(\Delta X_2)^2 = \langle \hat{X}_2^2 \rangle (t) = \frac{1}{4} \Big[\cosh r \sinh r \cos(2\omega t) + 1 + 2\sinh^2 r \Big]$$

$$\therefore (\Delta X_1)^2 = a - b\cos(2\omega t)$$

$$(\Delta X_2)^2 = a + b\cos(2\omega t)$$



$e^{i(\theta+\pi)} = -e^{i\theta}$

Q4.

(a)

$$\boxed{\hat{D}\,\hat{a}\,\hat{D}^{\dagger} = e^{\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}}\,\hat{a}\,e^{-(\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a})} = \hat{a} - \alpha}$$

$$[\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a},\ \hat{a}] = -\alpha$$

$$\boxed{\hat{D}\,\hat{a}^{\dagger}\,\hat{D}^{\dagger} = \hat{a}^{\dagger} - \alpha^{*}}$$

$$[\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a},\ \hat{a}^{\dagger}] = -\alpha^{*}[\hat{a}, \hat{a}^{\dagger}] = -\alpha^{*}$$

$$\boxed{\hat{S}\,\hat{a}\,\hat{S}^{\dagger} = \hat{a}\cosh r + \hat{a}^{\dagger} e^{i\theta}\sinh r}$$

$$\boxed{\hat{S}\,\hat{a}^{\dagger}\,\hat{S}^{\dagger} = \hat{a}^{\dagger}\cosh r + \hat{a}\, e^{-i\theta}\sinh r}$$

(b) $\hat{a}\,|0\rangle = 0$

$$\hat{D}\,\hat{S}\,\hat{a}\,|0\rangle = 0$$

$$\left(\hat{D}\hat{S}\,\hat{a}\,\hat{S}^{-1}\hat{D}^{-1}\right)\hat{D}\,\hat{S}\,|0\rangle = 0$$

$$\left(\hat{D}\hat{S}\,\hat{a}\,\hat{S}^{-1}\hat{D}^{-1}\right)|\xi\rangle = 0$$

Now $\hat{D}\,\underline{\hat{S}\,a\,\hat{S}^{\dagger}}\,\hat{D}^{\dagger}$

$$= \hat{D}\left(\hat{a}\cosh r + \hat{a}^{\dagger} e^{i\theta}\sinh r\right)\hat{D}^{\dagger}$$

$$= (\hat{a} - \alpha)\cosh r + (\hat{a}^{\dagger} - \alpha^{*})\, e^{i\theta}\sinh r$$

$$= (\hat{a} - \alpha)\mu + (\hat{a}^{\dagger} - \alpha^{*})\,\nu$$

Act on $|\xi\rangle$ to give 0.

let $|\xi\rangle = \sum_n c_n |n\rangle$

$n+1 = m$ $\quad$ $n+1 = m$

$n = m-1$

$$\sum c_n (\hat{a} - \alpha)\mu |n\rangle + \sum c_n (\hat{a}^\dagger - \alpha^*)\nu |n\rangle = 0$$

$$\mu \sum c_n \hat{a} |n\rangle - \alpha\mu \sum c_n |n\rangle + \nu \sum c_n \hat{a}^\dagger |n\rangle - \alpha^* \nu \sum c_n |n\rangle = 0$$

$$\mu \sum_{n=1}^{\infty} c_n \sqrt{n} |n-1\rangle + \nu \sum_{n=0}^{\infty} c_n \sqrt{n+1} |n+1\rangle - (\alpha\mu + \alpha^*\nu) \sum_{n=0}^{\infty} c_n |n\rangle = 0$$

$$\mu \sum_{m=0}^{\infty} c_{m+1} \sqrt{m+1} |m\rangle + \nu \sum_{m=1}^{\infty} c_{m-1} \sqrt{m} |m\rangle - (\alpha\mu + \alpha^*\nu) \sum_{n=0}^{\infty} c_n |n\rangle = 0$$

$$\mu \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1} |n\rangle + \nu \sum_{n=1}^{\infty} c_{n-1} \sqrt{n} |n\rangle - (\alpha\mu + \alpha^*\nu) \sum_{n=0}^{\infty} c_n |n\rangle = 0$$

$$\sum_{n=1}^{\infty} \left(\mu c_{n+1} \sqrt{n+1} + \nu c_{n-1} \sqrt{n} - (\alpha\mu + \alpha^*\nu) c_n \right) |n\rangle$$

$$+ \quad c_1 |0\rangle - (\alpha\mu + \alpha^*\nu) |0\rangle = 0$$

$$\Rightarrow \boxed{c_1 = \alpha\mu + \alpha^*\nu} \overset{(\text{say})}{=} \gamma$$

and

$$\mu c_{n+1} \sqrt{n+1} = -\nu c_{n-1} \sqrt{n} + \gamma c_n$$

$$\boxed{c_{n+1} = -\frac{\nu}{\mu} \frac{c_{n-1}}{} \frac{\sqrt{n}}{\sqrt{n+1}} + \frac{c_n \gamma}{\mu \sqrt{n+1}}}$$

$$\boxed{c_{n+1} = -\frac{\nu\sqrt{n}}{\mu\sqrt{n+1}} c_{n-1} + \frac{\gamma c_n}{\mu\sqrt{n+1}} \text{ where } \gamma = \alpha\mu + \alpha^*\nu}$$

$$\mu\sqrt{n+1}\; C_{n+1} - \gamma C_n + \nu\sqrt{n}\; C_{n-1} = 0$$

$$\text{Let } C_n = \mathcal{N}\left(\frac{e^{i\theta}\tanh r}{2}\right)^{n/2} \frac{1}{\sqrt{\cosh r}}\, f_n(x)$$

$$\frac{n-1}{2} = \frac{n}{2}$$

$$= \mathcal{N}\left(\frac{\nu}{2\mu}\right)^{n/2} \frac{1}{\sqrt{\mu}}\, f_n(x)$$

$$\mu\sqrt{n+1}\; \cancel{\mathcal{N}} \cancel{\left(\frac{\nu}{2\mu}\right)^{\frac{n}{2}}} \left(\frac{\nu}{2\mu}\right)^{1/2} \frac{1}{\sqrt{\mu}}\, f_{n+1}(x)$$

$$- \gamma\, \cancel{\mathcal{N}} \cancel{\left(\frac{\nu}{2\mu}\right)^{n/2}} \frac{1}{\sqrt{\mu}}\, f_n(x)$$

$$+ \nu\sqrt{n}\; \cancel{\mathcal{N}} \cancel{\left(\frac{\nu}{2\mu}\right)^{n/2}} \left(\frac{\nu}{2\mu}\right)^{-1/2} \frac{f_{n-1}(x)}{\sqrt{\mu}} = 0.$$

$$\cancel{\sqrt{\mu}}\sqrt{n+1}\sqrt{\frac{\nu}{2\cancel{\mu}}}\, f_{n+1}(x) - \gamma\frac{1}{\sqrt{\mu}} f_n(x) + \frac{\nu\sqrt{n}}{\sqrt{\mu}}\sqrt{\frac{2\mu}{\nu}}\, f_{n-1}(x) = 0$$

$$\sqrt{n+1}\sqrt{\frac{\nu}{2}}\, f_{n+1}(x) - \gamma\frac{1}{\sqrt{\mu}} f_n(x) + \sqrt{\nu}\sqrt{n}\sqrt{\frac{2\mu}{\mu}}\, f_{n-1}(x) = 0$$

$$\sqrt{n+1}\, f_{n+1}(x) - \gamma\frac{1}{\sqrt{\mu}}\sqrt{\frac{2}{\nu}}\, f_n(x) + \frac{\cancel{\sqrt{\nu}}\sqrt{n}\sqrt{2\mu}}{\sqrt{\mu}}\sqrt{\frac{2}{\cancel{\nu}}}\, f_{n-1}(x) = 0$$

$$\cancel{2\sqrt{n}\sqrt{\mu}\; f_{n-1}(x) = 0}$$

$$\sqrt{n+1}\, f_{n+1}(x) - \gamma\sqrt{\frac{2}{\mu\nu}}\, f_n(x) + 2\sqrt{n}\, f_{n-1}(x)$$

$$\sqrt{\frac{2}{\mu\nu}} = \sqrt{\frac{2}{e^{i\theta}\sinh\lambda\cosh\lambda}} = \sqrt{2}\sqrt{\frac{2}{e^{i\theta}\sinh 2\lambda}} = 2\left(e^{i\theta}\sinh 2\lambda\right)^{-1/2}$$

The desired recursion relation is

$$\boxed{\sqrt{n+1}\, f_{n+1}(x) - 2\gamma\left(e^{i\theta}\sinh 2\lambda\right)^{-1/2} f_n(x) + 2\sqrt{n}\, f_{n-1}(x) = 0} \quad (1)$$

Recursive relation for Hermite polynomials is:

$$H_{n+1}(x) - 2x\, H_n(x) + 2n\, H_{n-1}(x) = 0$$

$$f_n(x) = \frac{H_n(x)}{\sqrt{n!}}, \qquad x = \gamma\left(e^{i\theta}\sinh 2\lambda\right)^{-1/2}$$

$\sqrt{n+1}$ Eq(1) becomes

$$\sqrt{n+1}\,\frac{H_{n+1}(x)}{\sqrt{(n+1)!}} - 2x\,\frac{H_n(x)}{\sqrt{n!}} + 2\sqrt{n}\,\frac{H_{n-1}(x)}{\sqrt{(n-1)!}} = 0$$

$$\frac{\cancel{\sqrt{n+1}}}{\sqrt{n!}\,\cancel{\sqrt{(n+1)}}}\, H_{n+1}(x) - 2x\,\frac{H_n(x)}{\sqrt{n!}} + \frac{2\sqrt{n}}{\sqrt{(n-1)!}}\, H_{n-1}(x) = 0$$

$$H_{n+1}(x) - 2x\, H_n(x) + \frac{2\sqrt{n}\,\cancel{\sqrt{n!}}}{\cancel{(n-1)\sqrt{n!}}\,(n-1)!}\, H_{n-1}(x) = 0$$

$$\boxed{H_{n+1}(x) - 2x\, H_n(x) + 2n\, H_{n-1}(x) = 0} \quad (2)$$

(e)

$$\boxed{|\psi\rangle = \mathcal{N}\sum_{n=0}^{\infty}\left(\frac{e^{i\theta}\tanh r}{2}\right)^{n/2}\frac{1}{\sqrt{\cosh r}}\frac{1}{\sqrt{n!}}H_n\left(\gamma\left(\frac{e^{i\theta}\tanh r}{2}\right)^{1/2}\right)}$$

where $\gamma = \alpha\mu + \alpha^{*}\nu$

$= \alpha\cosh r + \alpha^{*}e^{i\theta}\sinh r.$

Find $\mathcal{N}$:

$\cancel{\langle n|\psi\rangle}$

$$C_0 = \mathcal{N}\frac{1}{\sqrt{\cosh r}}\left(\frac{H_0(x)}{\sqrt{0!}}\right)^{\to 1} = \frac{\mathcal{N}}{\sqrt{\cosh r}}.$$

Now $|\psi\rangle = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle$

$$C_0 = \langle 0|\psi\rangle = \langle 0|\hat{D}(\alpha)\hat{S}(\xi)|0\rangle = \langle -\alpha|\hat{S}(\xi)|0\rangle$$

$\hat{D}(\alpha)|0\rangle = |\alpha\rangle$

$\hat{D}(-\alpha)|0\rangle = |-\alpha\rangle$

$\hat{D}^{\dagger}(\alpha)|0\rangle = |-\alpha\rangle$

$\langle 0|\hat{D}^{\dagger}(\alpha) = \langle -\alpha|$

squeezed vacuum state.

$C_0 = \langle -\alpha|\xi\rangle$ $\cancel{\neq \frac{1}{\sqrt{\cosh r}}}$

$$= \langle -\alpha | \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} (-1)^n \left(e^{i\theta} \tanh r\right)^n \sqrt{\frac{(2n-1)!!}{(2n)!!}} \, |2n\rangle$$

$$= \langle -\alpha | \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} (-1)^n \left(e^{i\theta} \tanh r\right)^n \sqrt{\frac{(2n)!}{2^n n!}} \, |2n\rangle$$

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha)^n}{\sqrt{n!}} |n\rangle$$

$$\langle \alpha | = e^{-|\alpha|^2/2} \sum \frac{(\alpha^*)^n}{\sqrt{n!}} \langle n|$$

$$\langle -\alpha | = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(-\alpha^*)^n}{\sqrt{n!}} \langle n|$$

$$\therefore C_0 = \frac{e^{-|\alpha|^2/2}}{\sqrt{\cosh r}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-\alpha^*)^m}{\sqrt{m!}} (-1)^n \left(e^{i\theta} \tanh r\right)^n \frac{\sqrt{(2n)!}}{2^n n!} \underbrace{\langle m | 2n\rangle}_{\delta_{m,2n}}$$

$$C_0 = \frac{e^{-|\alpha|^2/2}}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{(-\alpha^*)^{2n}}{\cancel{\sqrt{(2n)!}}} (-1)^n \left(e^{i\theta} \tanh r\right)^n \frac{\cancel{\sqrt{(2n)!}}}{2^n n!}$$

$$\therefore \boxed{\begin{aligned} \mathcal{N} &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha^*)^{2n} (-1)^n}{2^n n!} \left(e^{i\theta} \tanh r\right)^n \\ &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \left(\frac{-(\alpha^*)^2 e^{i\theta} \tanh r}{2}\right)^n \frac{1}{n!} \end{aligned}}$$

$$\mathcal{N} = e^{-|\alpha|^2/2} \; e^{-\frac{(\alpha^*)^2 e^{i\theta}\tanh r}{2}}$$

$$\therefore \quad \boxed{|\psi\rangle = \exp\left(-\frac{|\alpha|^2}{2} - \frac{1}{2}(\alpha^*)^2 e^{i\theta}\tanh r\right) \times \sum_{n=0}^{\infty}\left(\frac{e^{i\theta}\tanh r}{2}\right)^{n/2}\frac{1}{\sqrt{\cosh r}}\frac{1}{\sqrt{n!}} H_n\left(\gamma\left(e^{i\theta}\sinh 2r\right)^{-1/2}\right)}$$

$$\cancel{P(n) = \exp\left(-|\alpha|^2 - (\alpha^*)^2 e^{i\theta}\tanh r\right) \times \sum_{n=0}^{\infty}\left(\frac{\tanh r}{2}\right)^{n}\frac{1}{\cosh r}\frac{1}{n!}\left|H_n\left(\gamma\left(e^{i\theta}\sinh 2r\right)^{-1/2}\right)\right.}$$

Plot. $\boxed{\bar{n} = |\alpha|^2 = 50,}$ $|\alpha| = \sqrt{50}$, $r = 0.5$, θ

$$\alpha = \sqrt{50}\, e^{i\varphi}, \quad \varphi = \theta/2$$

$$\varphi = \frac{\pi}{2} + \frac{\theta}{2}$$

$$\langle n|\psi\rangle = \exp\left(-\frac{|\alpha|^2}{2} - \frac{1}{2}(\alpha^*)^2 e^{i\theta}\tanh r\right)\left(e^{i\theta}\frac{\tanh r}{2}\right)^{n/2}\frac{1}{\sqrt{\cosh r}}\frac{1}{\sqrt{n!}} H_n$$

$$|\langle n|\psi\rangle|^2 = P(n) = \exp\left(-|\alpha|^2\right)\exp\left(-\frac{1}{2}(\alpha)^2 e^{-i\theta}\tanh r - \frac{1}{2}(\alpha^*)^2 e^{2i\theta}\right.$$

$$\left(\frac{\tanh r}{2}\right)^{n}\frac{1}{(\cosh r)}\frac{1}{n!}\left|H_n\left(\gamma\left(e^{i\theta}\sinh 2r\right)\right.\right.$$

$$P(n) = \exp(-|\alpha|^2)\exp\left(-\frac{\tanh r}{2}\left(\alpha^2 e^{-i\theta} + (\alpha^*)^2 e^{i\theta}\right)\right)$$
$$\times \left(\frac{\tanh r}{2}\right)^n \frac{1}{\cosh r}\frac{1}{n!}\left|H_n\left(\gamma(e^{i\theta}\sinh 2r)^{-1/2}\right)\right|^2$$

$$\alpha = |\alpha| e^{i\psi}$$

$$(\alpha)^2 = |\alpha|^2 e^{2i\psi}$$

$$(\alpha^*)^2 = |\alpha|^2 e^{-2i\psi}$$

$$\alpha^2 e^{-i\theta} = |\alpha|^2 e^{i(2\psi-\theta)}$$

$$(\alpha^*)^2 e^{i\theta} = |\alpha|^2 e^{-i(2\psi-\theta)}$$

$$P(n) = \exp(-|\alpha|^2)\exp\left(-\tanh r\,|\alpha|^2 \overbrace{\cos(2\psi-\theta)}^{\pm 1}\right)$$
$$\times \left(\frac{\tanh r}{2}\right)^n \frac{1}{\cosh r}\frac{1}{n!}\left|H_n\left(\gamma(e^{i\theta}\sinh 2r)^{-1/2}\right)\right|^2$$

$\psi = \theta/2$, Let $\theta = \frac{\pi}{3} = 60^\circ$

$\cancel{2\psi}$ $\psi = \frac{\pi}{6} = 30^\circ$.

$$(e^{-i\theta})^{1/2} \quad (e^n)^2 = e^{2n}$$