HW6

Q1.

→ Transforming operator is $\underbrace{e^{+i\hbar\omega_A' \hat{a}_A^\dagger \hat{a}_A t}}_{\hat{U}_A^\dagger} \underbrace{e^{+i\hbar\omega_B' \hat{a}_B^\dagger \hat{a}_B t}}_{\hat{U}_B^\dagger} = \hat{U}^\dagger(t)$.

→ In the interaction frame

$$\tilde{\hat{H}} = \hat{U}^\dagger \hat{H} \hat{U} - \hat{H} = -\hbar g\left(\hat{U}_A^\dagger (\hat{a}_A - \hat{a}_A^\dagger)\hat{U}_A\right)\left(\hat{U}_B^\dagger (\hat{a}_B - \hat{a}_B^\dagger)\hat{U}_B\right)$$

→ After the RWA in the interaction picture:

$$\tilde{\hat{H}} = +\hbar g\left[\hat{a}_A \hat{a}_B^\dagger e^{-i\hbar(\omega_A' - \omega_B')t} + \hat{a}_A^\dagger \hat{a}_B e^{+i\hbar(\omega_A' - \omega_B')t}\right]$$

$$= \hbar g\left(\hat{a}_A \hat{a}_B^\dagger e^{-i\Delta t} + \hat{a}_A^\dagger \hat{a}_B e^{+i\Delta t}\right) \quad ; \quad \Delta = \omega_A' - \omega_B'.$$

→ TDSE:

$$i\hbar \dot{c}_A(t)|10\rangle + i\hbar \dot{c}_B(t)|01\rangle$$

$$= \hbar g\left(c_A e^{-i\Delta t}\hat{a}_A \hat{a}_B^\dagger |10\rangle + c_A e^{+i\Delta t}\hat{a}_A^\dagger \hat{a}_B|10\rangle\right.$$

$$\left. + c_B e^{-i\Delta t}\hat{a}_A \hat{a}_B^\dagger|01\rangle + c_B e^{+i\Delta t}\hat{a}_A^\dagger \hat{a}_B|01\rangle\right)$$

$$= \hbar g\left(c_A(t) e^{-i\Delta t}|\cancel{10}\,01\rangle + c_B(t) e^{+i\Delta t}|10\rangle\right)$$

Projections:

$$\boxed{\begin{aligned} i\hbar \dot{c}_A(t) &= \hbar g e^{+i\Delta t} c_B(t) \\ i\hbar \dot{c}_B(t) &= \hbar g e^{-i\Delta t} c_A(t) \end{aligned}}\quad \begin{aligned} &—(1) \\ &—(2) \end{aligned}$$

Solve ① and ② simultaneously. Differentiating ①:

$$i\hbar\,\ddot{c}_A(t) = i\Delta\hbar g\, e^{i\Delta t} c_B(t) + \hbar g\, e^{i\Delta t}\dot{c}_B(t)$$

From ②

$$\dot{c}_B(t) = -\frac{i}{\cancel{\hbar}}\cancel{\hbar} g\, e^{-i\Delta t} c_A(t)$$

Insert above

$$i\hbar\,\ddot{c}_A(t) = i\Delta\hbar g\, e^{i\Delta t} c_B(t) + \hbar g^2 \cancel{e^{i\Delta t}}(-i)\cancel{e^{-i\Delta t}} c_A(t)$$

Inserting $c_B(t) = \frac{i\dot{c}_A(t)}{g} e^{-i\Delta t}$ from ①

$$i\hbar\,\ddot{c}_A(t) = i\Delta\hbar g\, \cancel{e^{i\Delta t}}\left(\frac{i\dot{c}_A(t)}{\cancel{g}}\cancel{e^{-i\Delta t}}\right) - i\hbar g^2 c_A(t)$$

$$i\hbar\,\ddot{c}_A(t) = -\Delta\hbar\,\dot{c}_A(t) - i\hbar g^2 c_A(t)$$

$$\ddot{c}_A(t) = i\Delta\,\dot{c}_A(t) - g^2 c_A(t)$$

$$\boxed{\ddot{c}_A(t) - i\Delta\,\dot{c}_A(t) + g^2 c_A(t) = 0}$$

Solution is

$$c_A(t) = e^{\frac{i\Delta t}{2}}\left(A e^{\frac{i\sqrt{\Delta^2+4g^2}}{2}t} + B e^{-\frac{i\sqrt{\Delta^2+4g^2}}{2}t}\right)$$

$$\dot{c}_A(t) = e^{\frac{i\Delta t}{2}}\left(i\left(\frac{\Delta}{2} + \frac{\sqrt{\Delta^2+4g^2}}{2}\right)\left(A e^{\frac{i\sqrt{\Delta^2+4g^2}}{2}t}\right) + i\left(\frac{\Delta}{2} - \frac{\sqrt{\Delta^2+4g^2}}{2}\right) B e^{-\frac{i\sqrt{\Delta^2+4g^2}}{2}t}\right.$$

From (1)

$$C_B(t) = \frac{i}{g} e^{-i\Delta t} \dot{c}_A(t)$$

$$\therefore C_B(t) = -\frac{1}{g}\left[\left(\frac{\Delta}{2} + \frac{\sqrt{\Delta^2+4g^2}}{2}\right) A e^{i\frac{\sqrt{\Delta^2+4g^2}}{2}t} + \left(\frac{\Delta}{2} - \frac{\sqrt{\Delta^2+4g^2}}{2}\right) B e^{-i\frac{\sqrt{\Delta^2+4g^2}}{2}t}\right]$$

At $t=0$, $C_B(t) = 0$, $C_A(t) = 1$, imply the following:

$\boxed{1 = A + B}$ and

$$0 = A\left(\frac{\Delta}{2} + \frac{\sqrt{\Delta^2+4g^2}}{2}\right) + B\left(\frac{\Delta}{2} - \frac{\sqrt{\Delta^2+4g^2}}{2}\right) = 0$$

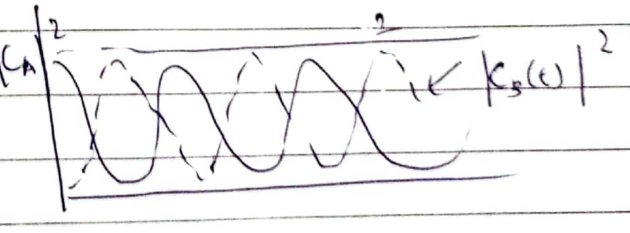
$$\boxed{\frac{\Delta}{2}(A+B) + \left(\frac{\sqrt{\Delta^2+4g^2}}{2}\right)(A-B) = 0}$$

→ On resonance: $\Delta = 0 \Rightarrow A = B$.

So $$C_B(t) = -\frac{1}{\cancel{g}}\left[\cancel{g} A e^{igt} - \cancel{g} A e^{-igt}\right]$$

$$= -A\left(e^{igt} - e^{-igt}\right) = -2iA\sin(gt).$$

And $C_A(t) = 2A\cos(gt)$



Hence Rabi flopping.

Q2

(a) From (3):

$$i\hbar \frac{1}{2\sqrt{n_1}} e^{i\theta_1} \frac{dn_1}{dt} + i\hbar (i) \sqrt{n_1} \frac{d\theta_1}{dt} e^{i\theta_1} = -ev\sqrt{n_1} e^{i\theta_1} + K\sqrt{n_2} e^{i\theta_2}$$

$$i\hbar e^{i\theta_1} \frac{dn_1}{dt} - 2\hbar n_1 \frac{d\theta_1}{dt} e^{i\theta_1} = -2ev n_1 e^{i\theta_1} + 2K\sqrt{n_1 n_2} e^{i\theta_2}$$

$$\boxed{i\hbar \frac{dn_1}{dt} - 2\hbar n_1 \frac{d\theta_1}{dt} = -2ev n_1 + 2K\sqrt{n_1 n_2}\, e^{-i\theta}} \quad \text{— (A)}$$

From (4):

$$\boxed{i\hbar \frac{dn_2}{dt} - 2\hbar n_2 \frac{d\theta_2}{dt} = 2ev n_2 + 2K\sqrt{n_1 n_2}\, e^{+i\theta}} \quad \text{— (B)}$$

where $\theta = \theta_1 - \theta_2$.

b) Re(A):

$$-2\hbar n_1 \frac{d\theta_1}{dt} = -2ev n_1 + 2K\sqrt{n_1 n_2} \cos\theta$$

$$\boxed{\frac{d\theta_1}{dt} = \frac{ev}{\hbar} - \frac{K}{\hbar}\sqrt{\frac{n_2}{n_1}} \cos\theta} \quad \text{— (C)}$$

Im(A) : $\hbar \dfrac{dn_1}{dt} = -2K\sqrt{n_1 n_2}\,\sin\theta$

$$\boxed{\frac{dn_1}{dt} = -\frac{2K}{\hbar}\sqrt{n_1 n_2}\,\sin\theta} \quad \text{— (D)}$$

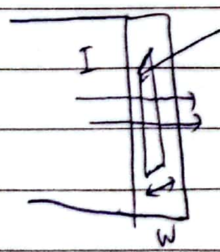
Re(B) : $-2\hbar n_2 \dfrac{d\theta_2}{dt} = 2ev\,n_2 + 2K\sqrt{n_1 n_2}\,\cos\theta$

$$\boxed{\frac{d\theta_2}{dt} = -\frac{ev}{\hbar} - \frac{K}{\hbar}\sqrt{\frac{n_1}{n_2}}\,\cos\theta} \quad \text{— (E)}$$

Im(B) : ~~2ħd~~ $\hbar\dfrac{dn_2}{dt} = 2K\sqrt{n_1 n_2}\,\sin\theta$

$$\boxed{\frac{dn_2}{dt} = \frac{2K}{\hbar}\sqrt{n_1 n_2}\,\sin\theta = -\frac{dn_1}{dt}} \quad \text{— (F)}$$

(C) $I_1 = -\dfrac{\partial}{\partial t}(-2en_1)\,Aw = 2e\left(\dfrac{\partial n_1}{\partial t}\right)Aw$



$$= 2e\left(-\frac{2K}{\hbar}\right)\sqrt{n_1 n_2}\,Aw\,\sin\theta$$

$\Rightarrow \boxed{I_1 = |I_c|\sin\theta}$ first JJ equ.

$$|I_c| = \left(\frac{4e^2K}{\hbar}\sqrt{n_1 n_2}\,Aw\right)$$

second JJ eqn.

(d) From (C) and (E):

$$\frac{d}{dt}(\theta_1-\theta_1) = \boxed{\frac{d\theta}{dt} = \frac{2ev}{\hbar}} \quad (n_1 = n_2)$$

(e)

$$I_1 = I_c \sin\theta$$

$$\frac{dI_1}{dt} = I_c \cos\theta \frac{d\theta}{dt}$$

$$= I_c \sqrt{1-\sin^2\theta}\left(\frac{2ev}{\hbar}\right)$$

$$\frac{dI_1}{dt} = I_c \sqrt{1-\left(\frac{I_1}{I_c}\right)^2}\left(\frac{2ev}{\hbar}\right)$$

$$v = \left(\frac{\hbar}{2ev I_c\sqrt{1-\left(\frac{I_1}{I_c}\right)^2}}\right)\frac{dI_1}{dt}$$

L_{eff} becomes a function of I_1, $L_{eff} = \frac{\hbar}{2evI_c\sqrt{1-\left(\frac{I_1}{I_c}\right)^2}}$

and non linear.

Q3

(a) $I_1 + I_2 = 0$ current entering a node = current leaving the node.

$$I_c \sin\theta + \frac{d}{dt}(C_J v) = 0$$

$$I_c \sin\theta + C_J \frac{dv}{dt} = 0$$

As $v = \frac{d\Phi}{dt}$ $\Rightarrow$ $\boxed{I_c \sin\theta + C_J \frac{d^2\Phi}{dt^2} = 0}$

(b) Second JJ eqn, part d) of Q2:

$$\frac{d\theta}{dt} = \frac{2e}{\hbar}\frac{d\Phi}{dt}$$

$$\frac{d\Phi}{dt} = \frac{\hbar}{2e}\frac{d\theta}{dt} \Rightarrow \frac{d^2\Phi}{dt^2} = \frac{\hbar}{2e}\frac{d^2\theta}{dt^2}$$

$$\boxed{I_c \sin\theta + C_J \frac{\hbar}{2e}\frac{d^2\theta}{dt^2} = 0} \quad \text{— (G)}$$

$$I_c \sin(2\pi \Phi/\Phi_0) + C_J \hbar/2e \, d^2\theta/dt^2 = 0$$

$$\frac{\hbar}{2e}\theta = \Phi$$

$$\theta = \frac{2e}{\hbar}\Phi$$

$$= \frac{2e}{h} 2\pi \Phi$$

$$= 2\pi \Phi/\Phi_0$$

(c) ~~$\mathcal{L}$ is unknown, but~~ $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\Phi}\right)$

Desired answer $\Rightarrow$ $\boxed{I_c \sin\left(2\pi \frac{\Phi}{\Phi_0}\right) + C_J \frac{d\dot\Phi}{dt} = 0}$ — (H)

$\mathcal{L}$ is unknown

$$I_c \sin\left(2\pi \frac{\Phi}{\Phi_0}\right) + C_J \frac{d\dot\Phi}{dt} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\Phi}\right) - \frac{\partial \mathcal{L}}{\partial \Phi}$$

This would imply $\frac{\partial \mathcal{L}}{\partial \Phi} = -I_c \sin(2\pi \Phi/\Phi_0)$

$$\mathcal{L}(\Phi, \dot{\Phi}) = \frac{\Phi_0}{2\pi} I_c \cos\left(2\pi \Phi/\Phi_0\right).$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\Phi}}\right) = C_J \frac{d}{dt}\dot{\Phi}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C_J \dot{\Phi} \Rightarrow \mathcal{L} = \frac{C_J \dot{\Phi}^2}{2}$$

For this question equation of motion is known, Lagrangian unknown

$$\Rightarrow \boxed{\mathcal{L}(\Phi, \dot{\Phi}) = \frac{I_c}{2\pi}\Phi_0 \cos\left(2\pi \Phi/\Phi_0\right) + \frac{C_J \dot{\Phi}^2}{2}}$$

d

$$Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C_J \dot{\Phi}$$

e

$$\boxed{H = \dot{\Phi} Q - \mathcal{L}}$$

$$H = C_J \dot{\Phi}^2$$

$$H = C_J \dot{\Phi}^2 - \frac{C_J \dot{\Phi}^2}{2} - \frac{I_c}{2\pi}\Phi_0 \cos\left(2\pi \Phi/\Phi_0\right)$$

$$\boxed{\hat{H} = \frac{1}{2} C_J \dot{\Phi}^2 - \frac{I_c}{2\pi}\Phi_0 \cos\left(2\pi \Phi/\Phi_0\right)}$$

From this point onward, everything is straightforward. Expand the cos factor in $\hat{H}$, substitute the ladder operators, and keep the energy preserving terms