Quantum Optics: HW6 by Muhammad Sabieh Anwar

Please submit before 10 am, Monday, 12 May 2025; only by the channel communicated

1. The Hamiltonian for two coupled harmonic oscillators is

$$\hat{H} = \hbar\omega'_A(\hat{a}_A^\dagger\hat{a}_A + \frac{1}{2}) + \hbar\omega'_B(\hat{a}_B^\dagger\hat{a}_B + \frac{1}{2}) - \hbar g(\hat{a}_A - \hat{a}_A^\dagger)(\hat{a}_B - \hat{a}_B^\dagger). \quad (1)$$

Transform this Hamiltonian to the interaction picture, indicating your transforming operators. Using the time-dependent Schrodinger equation, derive differential equations for the probability amplitudes $c_A(t)$ and $c_B(t)$ where the quantum state is given in its general form by,

$$|\psi(t)\rangle = c_A(t)|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B \quad (2)$$

and hence show that (on resonance) the system exhibits Rabi flopping, exchanging a single photon between the two oscillators.

2. The purpose of this question is to derive the DC Josephson equations. Consider the junction shown in the accompanying diagram.

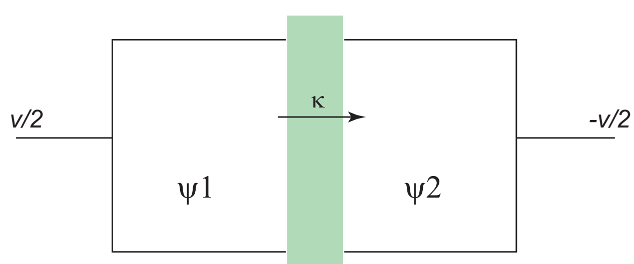

Figure 1: Josephson junction.

If e is the charge on each electron, we can write the time-independent Schrodinger equation for the two superconducting wavefunctions ψ_1

and ψ_2 as,

$$i\hbar \frac{d}{dt}\psi_1 = -ev\,\psi_1 + \kappa\,\psi_2 \tag{3}$$

$$i\hbar \frac{d}{dt}\psi_2 = ev\,\psi_2 + \kappa\,\psi_1 \tag{4}$$

where $\psi_1 = \sqrt{n_1}\,e^{i\theta_1}$ and $\psi_2 = \sqrt{n_2}\,e^{i\theta_2}$, and $2e$ is the charge on the Cooper pair.

(a) Using the trial wavefunctions, express Equations 3 and 4 as derivatives of the Cooper pair number densities n_1 and n_2 and derivatives of the order parameters θ_1 and θ_2.

(b) Separate out the real and imaginary parts of the foregoing equations to find expressions for $\frac{\partial\theta_1}{\partial t}$, $\frac{\partial n_1}{\partial t}$, $\frac{\partial\theta_2}{\partial t}$ and $\frac{\partial n_2}{\partial t}$. What is the relationship between $\frac{\partial n_1}{\partial t}$ and $\frac{\partial n_2}{\partial t}$?

(c) Using the continuity equation fof electric charge,

$$I_1 = \int \mathbf{J}_1 \cdot d\mathbf{A} = -\frac{\partial}{\partial t}\int \rho_1 dV \tag{5}$$

where $\rho_1 = n_1\,(-2e)$ is the charge density of Cooper pairs, derive the first DC Josephson equation

$$I_1 = I_c \sin\theta \tag{6}$$

where $\theta = \theta_1 - \theta_2$. What is the value of I_c?

(d) From this point onward, assume that the superconductors across the Josephson junction are identical, hence $n_1 = n_2$. Express $\frac{\partial\theta}{\partial t}$ in terms of the voltage v. This will furnish the second Josephson equation.

(e) Find the derivative of I_1 derived in part (d) and use this to determine the effective inductance of the junction. Is it a nonlinear function of I_1?

3. Consider the transmon shown.

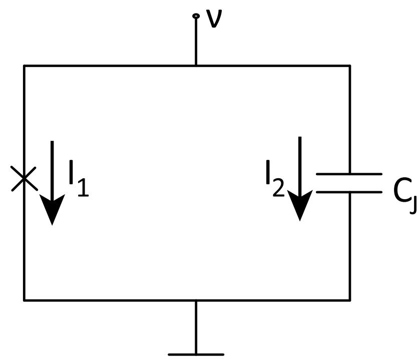

Figure 2: The transmonic loop.

Kirchoff's current law would mean $I_1 + I_2 = 0$.

(a) Using the two DC Josephson equations (e.g. derived from the previous question), show that the current law leads to the equation,

$$I_c \sin\theta + C_J \frac{d^2\Phi}{dt^2} = 0. \tag{7}$$

(b) Using the second Josephson equation, express θ in terms of Φ and replace this value of θ into Eq. 7.

(c) Equate this equation to the Euler-Lagrange equation and infer the Lagrangian for this system.

(d) From the Lagrangian find the charge variable $Q = \partial\mathcal{L}/\partial\dot{\Phi}$.

(e) Deduce the Hamiltonian for the Josephson junction, $\hat{H} = \dot{\Phi}Q - \mathcal{L}$. Your answer should be:

$$\hat{H} = \frac{\hat{Q}^2}{2C_J} - \frac{I_c\Phi_o}{2\pi}\cos(2\pi\frac{\hat{\Phi}}{\Phi_o}). \tag{8}$$

(f) Show that the Hamiltonian 8 can also be written as,

$$\hat{H} = 4E_c(\frac{\hat{Q}}{2e})^2 - E_J\cos(2\pi\frac{\hat{\Phi}}{\Phi_o}) \tag{9}$$

where $E_c = e^2/(2C_J)$ is the charging energy and $E_J = I_c\Phi_o/(2\pi)$ is the Josephson junction energy.

(g) With the identifications,

$$\begin{aligned} \hat{\Phi} &= \frac{\Phi_o}{2\pi}(\frac{2E_c}{E_J})^{1/4}(\hat{c} + \hat{c}^\dagger) && (10) \\ \hat{Q} &= -ie(\frac{E_J}{2E_c})^{1/4}(\hat{c} - \hat{c}^\dagger) && (11) \end{aligned}$$

cast the Hamiltonian 9 in terms of ladder operators. Expand the cosine function using the McLaurin series, preserve the particle conserving terms and simplify the Hamiltonian.

(h) What are the stationary energy levels?

(i) What are the separations between the energy levels? Between $|0\rangle$ and $|1\rangle$; and between $|1\rangle$ and $|2\rangle$?