Problem 1

Levenberg–Marquardt scheme

1. In the first part of this question, I would like you to generate an exponentially decaying data,
$$5e^{-0.2t}, \tag{1}$$
where $t \in [0, 10]$. Add random Gaussian noise so that you obtain a noisy decaying exponential. Plot this decay.

2. We would like to fit this fabricated data to an exponential decay,
$$x_1 e^{-x_2 t}, \tag{2}$$
where the unknown parameters are $\vec{x} = (x_1, x_2)$. Now I would like you to write a computer program employing the Levenberg-Marquardt scheme, which specifies that the parameters are updated according to
$$x_{k+1} = x_k - (H(x_k) + \beta \, \mathrm{diag}(H(x_k)))^{-1} \vec{\nabla}\chi^2. \tag{3}$$
Find χ^2, $\vec{\nabla}\chi^2$, and the Hessian $H(\vec{x})$, and where $\mathrm{diag}(H)$ is a fixed diagonal matrix with the same elements as H. Note that in the Levenberg-Marquardt algorithm, the value of β is also changed iteratively. Implement the following scheme.

   (a) Pick an initial guess for x, and $\beta = 0.01$ (say).

   (b) Find $\chi^2(\vec{x})$.

   (c) Compute $\delta\vec{x} = -(H + \beta \, \mathrm{diag}(H))^{-1}\vec{\nabla}\chi^2$.

   (d) Find $\chi^2(\vec{x} + \delta\vec{x})$.

   (e) If $\chi^2(\vec{x} + \delta\vec{x}) \geq \chi^2(\vec{x})$, increase β by a factor of 10 and go back to step (c).

   (f) If $\chi^2(\vec{x} + \delta\vec{x}) < \chi^2(\vec{x})$, decrease β by a factor of 10 and update the trial $\vec{x} \to \vec{x} + \delta\vec{x}$ and go to step (c).

   (g) Terminate if χ^2 does not change by a value greater than some threshold, e.g. 10^{-10} etc.

3. What is your best estimate of the parameter $\vec{x}$?

Problem 2

Question 7.5 from Hughes, "Measurements and their Uncertainties".

Problem 3

Question 7.7 from Hughes, "Measurements and their Uncertainties".

Problem 4

Question 7.8 from Hughes, "Measurements and their Uncertainties".

Problem 5

A student varies the orientation of a polarizer in steps of 10°, starting from 0° and going up to 270°. Polarized laser light passes through the polarizer and falls on a detector which measures the following intensities (in the same order and in arbitrary units).

$$\text{Intensity} = \{0.0558, -0.0274, -0.1127, -0.2253, -0.2098, -0.2725, -0.3086, -0.2611, -0.2681, -0.2607, -0.1492, -0.0891, -0.0157, 0.0506, 0.1276, 0.1229, 0.2285, 0.2074, 0.1596, 0.1071, 0.0166, -0.0053, -0.1177, -0.2067, -0.2031, -0.2898, -0.3173, -0.3006\}$$

I propose that a suitable fit for the data-set is

$$\text{Intensity} = A + B\cos^2(C\theta + D), \tag{4}$$

where θ is the polarizer angle in radians. Use simulated annealing to find the best estimate of $\{A, B, C, D\}$. Decrease the temperature, by say, 1% at each step of the iterative annealing process.