How water drains from a tank: understanding height and efflux velocity over time

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1 Background

Imagine you have a big water tank — like a bottle — with a small hole at the bottom. You fill the tank with water, and then you let it drain out from the hole (see Figure 1). A question you might ask is: how fast does the water level drop over time? Or, more precisely, can we find a formula for the height of the water $h(t)$ as time passes? Let's answer this by slowly building a model, using physical understanding and a bit of simple calculus. The tank has a constant cross-sectional area (it's a cylinder), which we'll call A. There's a small hole at the bottom, with a much smaller area, which we'll call a. Water flows out of this hole due to gravity. Now, let's think about the water height in the tank. If the water is deeper, it pushes harder on the hole at the bottom. This makes the water flow faster. When there is less water, the pressure is lower and the water flows more slowly. That's why the water drains quickly at first, but then more slowly later.
This idea was captured by a scientist named Torricelli. He found that the speed v of water flowing out of a hole at the bottom of a tank is.

$$v = \sqrt{2gh} \tag{1}$$

Here, g is the acceleration due to gravity (about $9.8\,\mathrm{m/s^2}$), h is the height of the water above the hole, and v is the speed of water coming out of the hole. Given that, volume of water in the tank is given by.

$$V(t) = A \cdot h(t) \tag{2}$$

This means that the volume V is equal to the cross-sectional area of the tank, A, multiplied by the height of the water column $h(t)$ at any time t. Here, A is the cross-sectional area of the tank, which is constant because the tank is cylindrical. As the water drains out of the tank, the volume decreases over time. The rate at which the volume changes is given by the derivative:

$$\frac{dV}{dt} = A \cdot \frac{dh}{dt} \tag{3}$$

This is our first use of a differential equation. It simply tells us how quickly the volume (and hence the height) of water is changing with respect to time. On the other hand, the water is flowing out through a small hole at the bottom of the tank. According to Torricelli's law, the speed v of water leaving the hole is given by $v = \sqrt{2gh}$. The flow rate, or volume of water leaving per unit time, is the area of the hole multiplied by the velocity.

$$\frac{dV}{dt} = -a \cdot v = -a \cdot \sqrt{2gh} \tag{4}$$

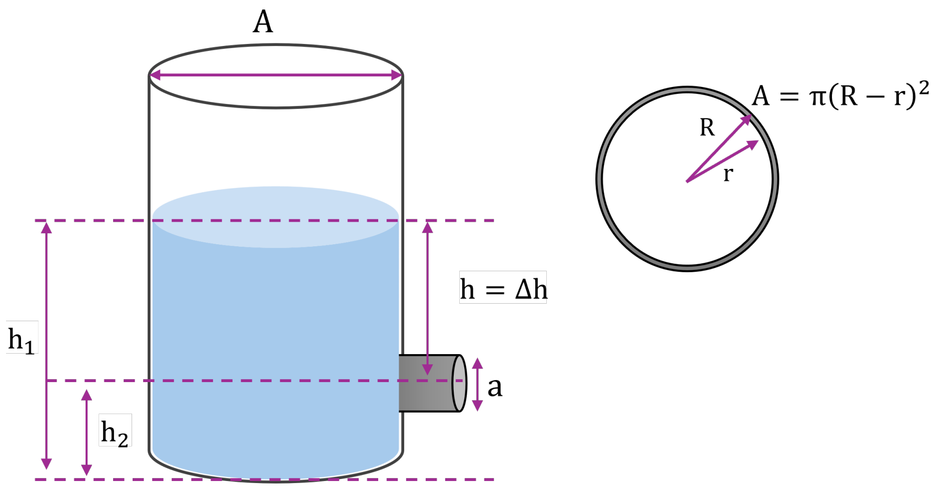

Figure 1: A cylinder with water flowing out from a narrow orifice at a fixed height from the base.

Here, a is the area of the small hole (orifice) at the bottom of the tank. The minus sign indicates that water is leaving the tank, so the volume is decreasing. We now can equate the two expressions for $\frac{dV}{dt}$:

$$A \cdot \frac{dh}{dt} = -a \cdot \sqrt{2gh} \tag{5}$$

This equation relates the rate of change of water height $\frac{dh}{dt}$ to the current water height h, and it forms the basis for solving the draining problem using differential equations. Solving the above results in attaining normalized height as a function of time as presented in equaiton 6, wherein $\tau = \frac{t}{T}$, T is the total time to drain the tank, and H is the initial height of water in the tank, $h' = \frac{h(t)}{H}$. The complete solution to attain the relation is provided in Appendix.

$$\boxed{h' = (1 - \tau')^2} \tag{6}$$

This is a universal result. It means, no matter what tank or orifice you use, height of the liquid decreases as the square of time, as long as you measure time as a fraction of the total drain time. This simple tells the full story of how the tank drains. Table 1 highlights how it plays out at specific moments in time.

Table 1: Theoretical water height as a function of normalized time

τ	$h'(t)$	Value	Explanation
0	$(1-0)^2$	1	Full tank
0.5	$(1-0.5)^2$	0.25	25% water remains
1	$(1-1)^2$	0	Empty tank

Students now see how a real-world problem turns into a differential equation, how to solve it step by step using calculus, and how to make the result universal by using simple substitutions.

We can now also estimate normalized efflux velocity, v' from the normalized height equation (6) and Torricelli equation (1).

$$\boxed{v' = (1 - \tau')} \tag{7}$$

The remainder of this report is organized as follows. Section 2 outlines the experimental setup and procedures used to measure water height and efflux velocity over time. Section 3 presents the experimental results. Finally, section 4 offers concluding remarks, summarizing the key findings.

2 Experiment

2.1 Setup

The experimental apparatus consists of a graduated cylinder with a small orifice at its base through which water discharges. Two versions of the experiment were performed to measure how the water height changes over time, as shown in Figure 2. In the setup shown in Figure 2a, the cylinder is mounted on a digital force sensor (PhysLoad), which is connected to a data acquisition device (PhysLogger). As water drains from the orifice into a collection container, the increasing mass of collected water is continuously recorded by PhysLoad. The PhysLogger software enables real-time data logging and visualization of the force (mass) measurements. This method provides an indirect but quantitative way to determine the height of the water column as a function of time.

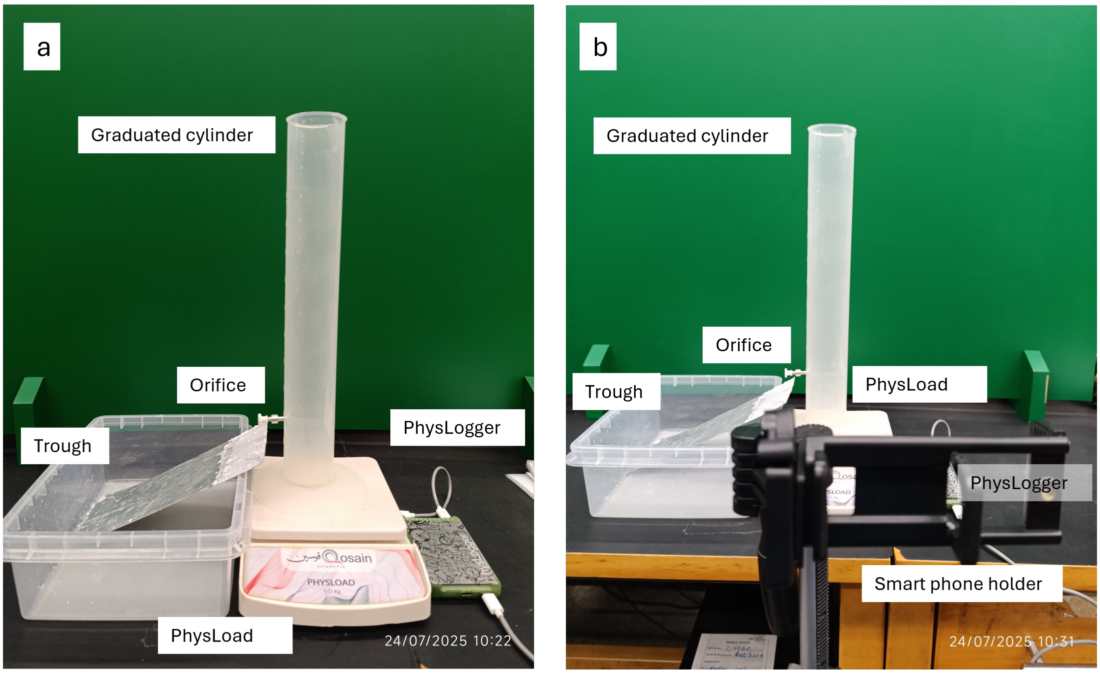

Figure 2: Two experimental configurations for observing draining dynamics of a water column. (a) Height is computed from mass data using PhysLogger and PhysLoad. (b) Height is directly tracked from video using Tracker software.

Another setup shown in Figure 2b, where a smartphone was used to record a video of the draining process. After recording, the video is imported into Tracker software, where the water surface level is tracked frame by frame. This allows direct measurement of the water height over time.
Further, in this setup, average diameter of the cylindrical tank is (49.49 ± 0.01) mm, and the diameter of the orifice is (24.70 ± 0.03) mm. The sampling frequency of the PhysLogger was set to 0.20 Hz (i.e., one sample every 50 seconds). The calibration stick used in the Tracker software is 10.1 cm in length, which was converted to meters (0.101 m) in the associated calibration step of the video analysis.

2.2 How to deduce height and efflux velocity over time from experimental approaches: PhysLoad and Tracker software?

As mentioned above, two experimental approaches were used to determine the height of water in a draining cylinder and the efflux velocity of water over time: one based on mass measurement using PhysLoad and the other on video analysis using Tracker.

2.2.1 PhysLoad: Mass based measurement

In this approach, the PhysLoad sensor records the mass of discharged water over time. The steps to extract height and velocity are as follows.

1. The mass, $m(t)$ is converted to volume using the relation.

$$V(t) = \frac{m(t)}{\rho}$$

   where ρ is the density of water (typically 1000 kg/m^3).

2. The water height in the cylinder is then found using the cylinder volume formula.

$$V = Ah \quad \Rightarrow \quad h(t) = \frac{V(t)}{A} = \frac{m(t)}{\rho A}$$

   where A is the cross-sectional area of the cylinder.

3. To find efflux velocity, we compute the mass flow rate, $\dot{m}$ using the midpoint (finite difference) method.

$$\dot{m}_k = -\frac{m_{k+1} - m_k}{t_{k+1} - t_k}$$

   This gives the rate at which mass exits the orifice at each time interval.

4. Finally, efflux velocity $v(t)$ is calculated using.

$$v(t) = \frac{\dot{m}(t)}{\rho \cdot a}$$

   where a is the area of the nozzle/orifice at the bottom of the cylinder.

2.2.2 Tracker: Video-based Measurement

In this approach, video frames are analyzed using Tracker software to obtain the water height, $h(t)$ directly over time.

1. The height data is used to infer volume using.

$$V(t) = A \cdot h(t)$$

2. The mass is then estimated from volume using.

$$m(t) = \rho \cdot V(t)$$

3. Once mass flow rate is computed, the same method described earlier is used to evaluate efflux velocity over time.

Both approaches ultimately allow us to analyze how the efflux velocity depends on water height, and to validate the Toricelli's law.

3 Results

3.1 Normalized height and velocity over time

Figure 3 (a) presents the time evolution of the normalized fluid height h/H and normalized efflux velocity v/V. providing comparison across different measurement methods: Tracker, and PhysLoad. The experimental data from Tracker align closely with the theoretical curve throughout the full drainage period, validating the use of video analysis in capturing the parabolic nature of height decay due to constant gravitational acceleration. For example, at $t/T = 0.5$, the theoretical

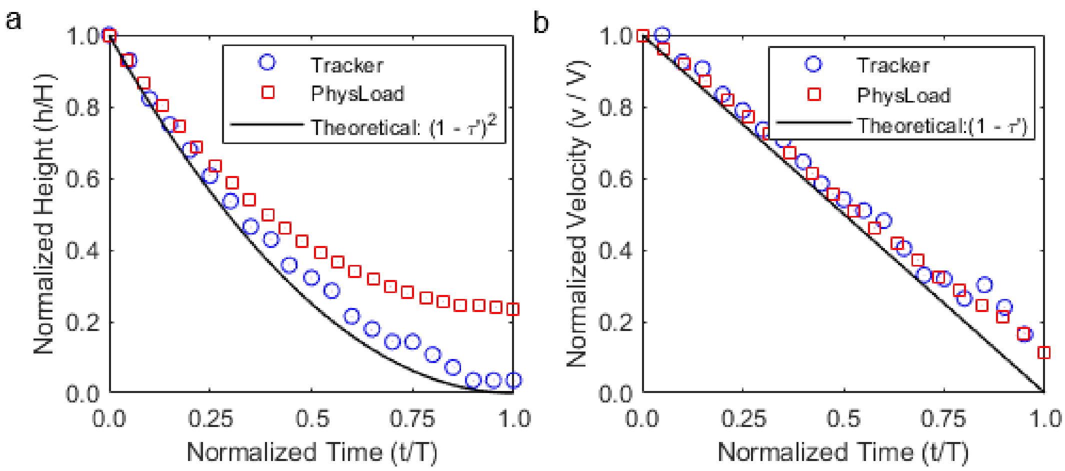

Figure 3: (a) Normalized height h/H vs. normalized time t/T, showing PhysLoad data alongside the theoretical prediction $h' = (1 - \tau')^2$. (b) Normalized efflux velocity V/V_0 vs. normalized time t/T, comparing PhysLoad measurements to the theoretical relation $V' = 1 - \tau'$.

normalized height is 0.25, and the Tracker value is approximately 0.32. At $t/T = 0.8$, Tracker gives a value of around 0.1, compared to the theoretical 0.04. However, the PhysLoad data results were slightly deviating. At $t/T = 0.55$, PhysLoad gives a value of about 0.52, which deviates by 56% from theoretical value, indicating lag precision. Figure 3(b) displays the normalized efflux velocity as a function of normalized time, with theoretical relation.Tracker data again exhibit strong consistency with the theoretical model, supporting the assumption that velocity decreases linearly as the fluid head reduces. At $t/T = 0.3$, the theoretical v/V is 0.7, and Tracker records the same 0.70. Even at $t/T = 0.9$, the Tracker remains close to 0.23, compared to theoretical 0.1. PhysLoad, however, reports significantly higher values at later times. At $t/T = 0.9$, it measures around 0.3, about three times greater than expected, indicating a growing deviation at lower heights.

3.2 Torricelli's law validation

To validate Torricelli's law, experimental data was plotted as velocity squared (v^2) versus height (h) in Figure 4, which implies a linear relationship between v^2 and h, with the slope representing $2g$. In Figure 4(a), Tracker data produce a best-fit linear relationship.

$$v^2 = 13.49 \cdot h - 0.15 \tag{8}$$

yielding an effective gravitational acceleration of $g \approx (6.74 \pm 0.29)\,\mathrm{m/s^2}$. Similarly, Figure 4(b) shows the PhysLoad results with the best-fit equation:

$$v^2 = 9.87 \cdot h - 0.61 \tag{9}$$

corresponding to $g \approx (5.12 \pm 0.41)\,\mathrm{m/s^2}$.

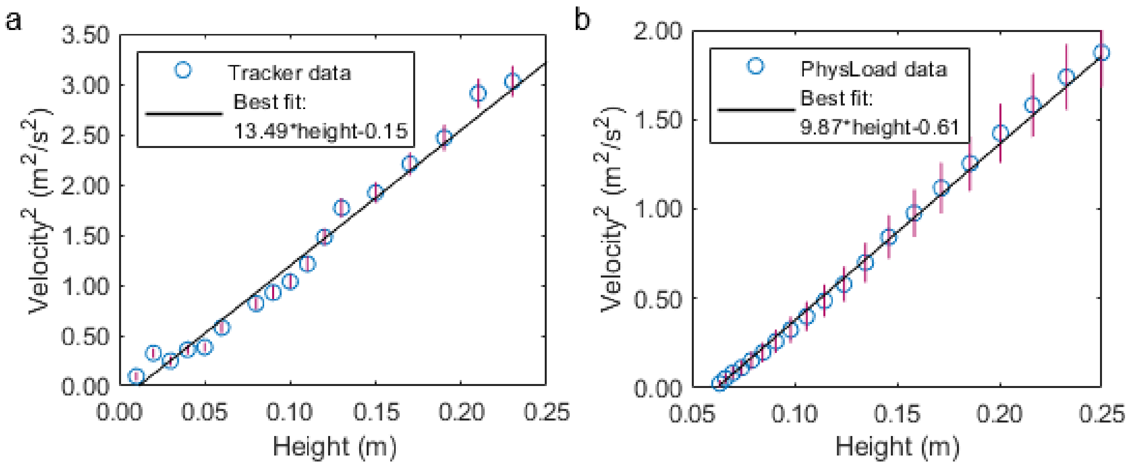

Figure 4: (a) Squared efflux velocity v^2 vs. water height h using Tracker. (b) Squared efflux velocity v^2 vs. water height h using PhysLoad.

While both methods confirm the linearity of v^2 with h, the experimental values of g are notably lower than the standard $9.86\,\mathrm{m/s^2}$. The possible reasons for this deviation include both experimental limitations and physical phenomena not captured in the idealized form of Torricelli's law. One key contributor is the formation of a *vena contracta*, a region just downstream of the orifice where the fluid jet narrows to its smallest cross-section due to the inertia of fluid particles and the convergence of streamlines as the fluid exits the nozzle [1]. In cases where the nozzle is non-uniform or tapering, the geometry itself promotes or enhances the formation of this *vena contracta* either within or just beyond the nozzle. This leads to a smaller effective flow area and consequently lower volumetric and exit velocities than those predicted by ideal models. Since Torricelli's equation assumes a complete conversion of potential energy to kinetic energy at the full orifice area, any reduction in actual velocity due to contraction results in an underestimation of g when experimental values are substituted back into the equation. Additionally, viscous effects, internal pressure losses, and the complex pressure gradient within a tapering nozzle further contribute to this discrepancy [1, 2] . These factors are typically accounted for using an empirical discharge coefficient, which incorporates both the velocity loss and the contraction effect. In practical, especially with irregular or tapering nozzle shapes discharge coefficient can be significantly less than 1, thereby explaining the observed deviation from theoretical expectations.

4 Conclusion

The experimental investigation of vertical efflux motion using both *Tracker* and *PhysLoad* tools validates the expected linear relationship between v^2 and height h, consistent with Torricelli's law. Tracker results are more close with theoretical predictions, suggesting higher accuracy in capturing motion through video analysis. However, both experimental methods yielded significantly lower values for gravitational acceleration g, approximately $6.74\,\mathrm{m/s^2}$ for Tracker and $5.12\,\mathrm{m/s^2}$ for PhysLoad, compared to the standard value of $9.81\,\mathrm{m/s^2}$. This deviation highlights the limitations of idealized models in describing real-world fluid behavior.The key contributor to this discrepancy is the formation of a *vena contracta*, a region of reduced flow area and increased streamline convergence just downstream of the orifice. This phenomenon, driven by inertial effects and nozzle geometry, results in a smaller effective outflow area and lower exit velocities than those predicted by the ideal model. Additional factors, such as viscous effects, internal pressure losses, and non-uniform nozzle shapes, further reduce the observed velocity. These real-world effects result in an underestimation of g when experimental velocities are substituted into the theoretical formula. To address this, empirical corrections—such as the discharge coefficient—are typically introduced to account for both velocity loss and contraction effects. In conclusion, while the experimental trends support the form of Torricelli's law, the quantitative results underscore the necessity of incorporating non-ideal effects to accurately model fluid efflux.

Appendix

The equation 5 in section 1 can be wriiten as,

$$\frac{dh}{dt} = -\frac{a}{A} \cdot \sqrt{2gh}$$

This is the key differential equation we will solve. It says that the rate of change of height depends on the square root of the current height.
We now solve this step by step. First, rewrite it in a form where we can integrate:

$$\frac{dh}{\sqrt{h}} = -\frac{a}{A}\sqrt{2g} \cdot dt$$

Now integrate both sides.

On the left side:

$$\int \frac{1}{\sqrt{h}}\, dh = 2\sqrt{h}$$

On the right side:

$$\int dt = t$$

So, putting it together:

$$2\sqrt{h} = -\frac{a}{A}\sqrt{2g} \cdot t + C$$

Here C is a constant that we'll figure out from the initial condition.

At time $t = 0$, the tank is full. Let's say the initial height is H. Plug that in:

$$2\sqrt{H} = C \quad \Rightarrow \quad C = 2\sqrt{H}$$

So the equation becomes:

$$2\sqrt{h} = -\frac{a}{A}\sqrt{2g} \cdot t + 2\sqrt{H}$$

Divide both sides by 2:

$$\sqrt{h} = \sqrt{H} - \frac{a}{2A}\sqrt{2g} \cdot t$$

Now square both sides to get rid of the square root:

$$h(t) = \left(\sqrt{H} - \frac{a}{2A}\sqrt{2g} \cdot t\right)^2$$

This gives us height as a function of time — but it's a bit messy. We want to make it cleaner and independent of the tank size or orifice size.

Let's define the total time T it takes to drain completely:

$$T = \frac{A}{a}\sqrt{\frac{2H}{g}}$$

Define a dimensionless time:

$$\tau = \frac{t}{T}$$

Define a dimensionless height:

$$h' = \frac{h(t)}{H}$$

Now substitute into our formula. After simplification, we get:

$$h' = (1 - \tau')^2$$

References

[1] Horsch, G. M. (2020). A simple model for the calculation of the fluid discharge from a small orifice. The Physics Teacher, 58(2), 113-115.

[2] D'Alessio, S. (2021). Torricelli's law revisited. European Journal of Physics, 42(6), 065808.