

Universalising Fluid Dynamics: A Study of the Vena Contracta and Residual Head in Water Discharge

In this experiment, the behaviour of water discharge through a pinhole is investigated and mathematised using the foundational principles of Fluid Dynamics. Using a large cylinder, Phys-load, and Phys-logger, we measured the *mass flow rate* to determine the relationship between the speed of the fluid (V) and the height (H). It was observed that the standard Torricelli's Law is very idealised, so we proposed a modified model that subtracts a residual head R to account for energy losses due to the vena contracta effect. The value of R was found to be approximately 0.050m, with a gradient of the squared speed versus height plot of 14.3 m/s². Followingly, we aimed to universalise these relationships by converting the variables into dimensionless quantities. The analysis revealed a linear relationship between the dimensionless speed, v , and the dimensionless time, t , confirming that the law holds true for any experiment, regardless of the apparatus used, despite the presence of viscosity and surface tension.

1. Introduction

1.1 Context and Motivation

The behaviour of water discharge through a pinhole is a classic problem in fluid dynamics, governed by conservation laws. However, textbook models often fail to predict real-world behaviour where energy dissipation occurs. Understanding these discharge rates is crucial for industrial chemical dosing, reservoir management, and irrigation systems that require precise flow control.

1.2 Objective

The primary objective of this experiment is to study the discharge of water from a pinhole and to mathematise these principles using Fluid Dynamics. Unlike standard verifications, this study aims to universalise these relationships through dimensionless analysis, ensuring they hold true for any apparatus. Specifically, we investigate the "novel" deviations caused by the vena contracta effect and surface tension at low fluid heights.

2. Theory

2.1 Bernoulli's Principle

Proposed by Daniel Bernoulli in 1738, Bernoulli's Principle is one of the most crucial and foundational models in the study of fluid dynamics. It is derived from the law of conservation of energy. It assumes the fluid to be ideal [1]. It is often simply represented as:

$$P + \frac{1}{2}\rho v^2 + \rho gh = \text{constant},$$

2.2 Torricelli's Law and the Modified Model

Though proposed before the Bernoulli equation itself, Torricelli's Law these days is derived from the Bernoulli equation. It is given by the expression,

$$v^2 = 2gH,$$

Where v is the speed of the fluid coming out of an orifice, g is the gravitational acceleration, and H is the depth of water above the orifice.

This form of the model, however, is very idealised; the fluid is assumed to be ideal, so the model does not account for energy losses due to turbulent flow. It also does not account for the phenomenon of vena contracta, as the orifice's diameter seems to be contracted, i.e., the effective area of the orifice is less than the actual measured area of the orifice.

Bernoulli's equation in this case will only hold true between two specific points when a head loss is subtracted from the height. To deal with this problem, in our experiment, we will be using a modified form of Torricelli's law:

$$v^2 = 2g(H - R),$$

Where all other variables are the same, R is a residual head, which is constantly subtracted from height, and in this way accounts for energy losses due to the vena contracta effect.

To calculate the offset value R , we need to plot the square of the range of the water projectile leaving the pinhole versus the height of the water in the container. The x-intercept of the fitted line for this scatter plot will give us the value of R . This is because range x is related to H as:

$$x = \sqrt{4H(H - R)},$$

On squaring both sides, we obtain a relation where, when plotting x^2 versus H , the x-intercept of the line of best fit represents an estimated value of R .

2.3 Equation of Continuity

The equation of continuity in general is an equation that describes the transport of some quantity. In fluid dynamics, the continuity equation states that the rate at which mass decreases inside the container is equal to the rate at which mass leaves the container, for an ideal fluid. This is represented by:

$$\frac{dm_{in}}{dt} = -\frac{dm_{out}}{dt},$$

Euler formalised the equation of continuity in his paper about fluid mechanics, which in simple terms is written as

$$\rho A_1 v_1 = \rho A_2 v_2,$$

In a closed pipe system, A_1 and v_1 are the area and velocity of fluid at one point, respectively, and A_2 and v_2 are the area and velocity of fluid at another point, respectively. This is also often written as $Av = \text{constant}$.

2.4 Dimensionless Quantities (Universalisation)

When some equation/idea is designated as a law by physicists, it is often imperative that the equation or idea is highly flexible. It shouldn't be that the law fits for one experiment and fails in another replication of the same experiment. To universalise a law, usually the technique of normalisation is used to make the involved quantities dimensionless.

Normalisation is done to scale the quantities between zero and one so that we can compare values of our experiment with perhaps a similar experiment. If the surface area of the pinhole in our experiment is significantly smaller than the surface area of the cylinder, we can make our quantities dimensionless. Dimensionless quantities are such that they are not dependent on varying external factors, i.e., in our experiment, when we use dimensionless quantities it does not matter what kind, size, or type of container/bottle/cylinder we use; if the law is correct, it holds true regardless.

Firstly, we need to make speed, V , dimensionless, which is simply done by dividing V with the maximum value of speed, V_0 :

$$v = \frac{V}{V_0},$$

Similarly for time, T , we simply divide T with the maximum value of time, T_0 :

$$t = \frac{T}{T_0},$$

[1] An ideal/ perfect fluid is a fluid that can be completely characterized by its rest frame mass density and isotropic pressure. Specifically, perfect fluids have no shear stresses, viscosity, or heat conduction

2. Theory

To make height, H , dimensionless, we need to take the ratio of H and the maximum height, H_0 , however we also need to subtract the head loss, R from each. So dimensionless quantity for height, h , becomes:

$$h = \frac{H - R}{H_0 - R},$$

To write Torricelli's law in terms of the dimensionless quantities, let's say we have speed V at height H , and speed V_0 at H_0 :

$$V^2 = 2g(H - R) \quad \text{--(i)},$$

$$V_0^2 = 2g(H_0 - R) \quad \text{--(ii)},$$

Dividing (i) by (ii):

$$\frac{V^2}{V_0^2} = \frac{H - R}{H_0 - R} \Rightarrow v^2 = h \Rightarrow v = \sqrt{h},$$

To derive another relation, if we implicitly differentiate Torricelli's equation, we get:

$$2V \frac{dV}{dT} = 2g \frac{dH}{dT},$$

Also, according to the continuity relation:

The left-hand side of the equation is for the flow of fluid in the cylinder, and the right-hand side of the equation is for the flow of fluid from the pinhole. If we substitute dH/dt , then we get :

$$\frac{dH}{dT} A = aV \Rightarrow \frac{dV}{dT} = \frac{ag}{A},$$

By solving the separable differential equation, we get

$$V = \frac{ag}{A}T + V_0,$$

We find the integration constant as V_0 , because at time zero, the speed of ejection is maximum. So, the equation becomes:

$$V = \frac{ag}{A}T + V_0,$$

This can be rewritten in terms of T_0 :

$$0 = \frac{ag}{A}T_0 + V_0 \Rightarrow V_0 = -\frac{ag}{A}T_0 \Rightarrow T_0 = -\frac{A}{ag}V_0,$$

Now that we have this expression for maximum time, going back to the original differential equation, divide both sides of the equation with V_0 :

$$\frac{V}{V_0} = \frac{ag}{AV_0}T + 1,$$

This can be rewritten as:

$$v = -\frac{1}{T_0}T + 1 \Rightarrow v = 1 - t,$$

So, now we have a relationship between the dimensionless quantities v and t . Followingly, because

$$v = \sqrt{h} \mapsto h = (1 - t)^2.$$

We have universalised Torricelli's Law, and when we plot our data for the dimensionless quantities, we expect that v and t will have a linear relationship, while h and t will have a quadratic relationship.

3. Experimental Setup and Method

Apparatus:

- A large cylinder with a pinhole
- Phys-load (Force Sensor)
- Phys-Logger and a computer
- An empty container to collect the water

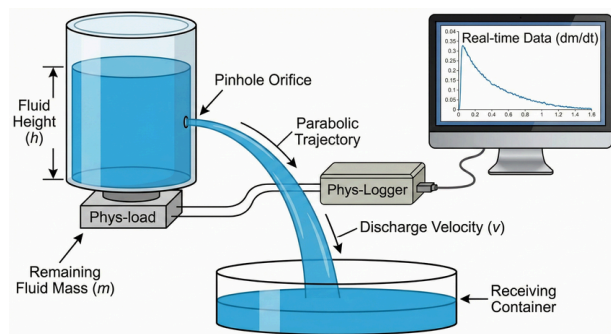


Fig 1: Schematic of the experimental apparatus. A cylindrical reservoir with a lateral pinhole orifice is positioned atop a Phys-load force sensor, which continuously records the mass of the remaining fluid (m) as it drains. The sensor is interfaced with a computer via a Phys-Logger to acquire real-time data (dm/dt). The discharged fluid follows a parabolic trajectory into a receiving container, allowing for the simultaneous calculation of discharge velocity and fluid height.

Procedure:

- The measurements that we need to make before starting the experiment are the diameter of the cylinder and the diameter of the pinhole—which will be used to calculate their respective surface areas.
- We will first tare the Phys-load by placing the empty cylinder on it—through the Phys-load taring option in the Phys-logger application—then fill the cylinder with normal tap water.
- Using Phys-logger, we will measure the different values of masses at different times. This will help us to find the mass flow rate by finding the derivative at each data point.

- Followingly, considering speed of the fluid to be V , we know by applying the equation of continuity at the pinhole:

$$\rho aV = \frac{dm}{dT},$$

- After some rearrangement, we get the expression to calculate V , for each data point:

$$V = \frac{dm}{dT} \frac{1}{\rho a},$$

- By applying the equation of continuity to the cylinder:

$$\frac{dH}{dT} \rho A = \frac{dm}{dT},$$

- After some rearrangement, we get the expression which we will use to calculate the height of the fluid, H , for each data point:

$$H = \frac{m}{\rho A}.$$

4. Data Analysis & Uncertainties

4.1 Error Analysis

One of the major causes of deviations between the predicted values and our calculated ones is due to the assumption that the fluid behaves as an ideal fluid; fluid is assumed to be incompressible, to have laminar flow, and have negligible viscosity. This is often not the case for real fluids; therefore, our values are only very close to what is otherwise predicted.

There are uncertainties in the values measured, i.e., in the measured values of diameters of the cylinder and the pinhole, then in the calculated values of projectile range, speed, and height. These same uncertainties are also carried forward into dimensionless quantities, and we do their appropriate analysis in MATLAB, to be able to plot error bars, wherever appropriate.

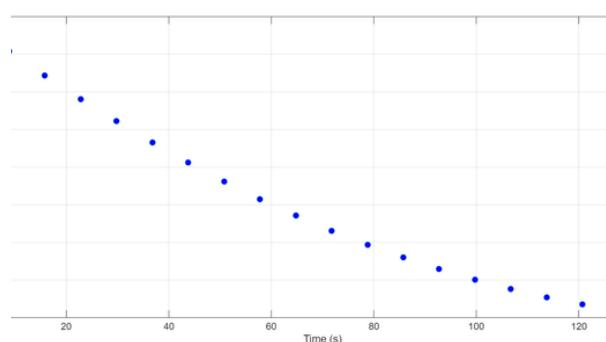


Figure 2: Temporal variation of fluid mass within the cylinder recorded via Phys-load. The decreasing magnitude of the slope (dm/dt) indicates a reduction in mass flow rate as the hydrostatic pressure diminishes over time.

5. Results

5.1 General Relationships

We use our readings to verify the different relationships by plotting:

- Mass vs Time
- Speed vs Time
- Speed vs Height
- Speed-squared vs Height

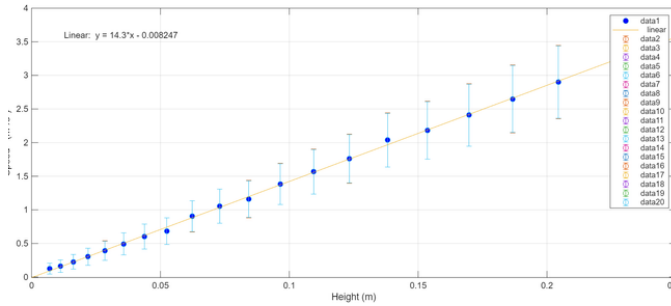


Figure 3: Linearization of Torricelli's Law (v^2 vs. H). The gradient of the best-fit line is 14.3 m/s^2 . This value represents $2g_{\text{effective}}$, which is significantly lower than the theoretical $2g_{\text{approx}}$ which is 19.6 m/s^2 , verifying the presence of substantial continuous energy losses.

5.2 Residual Head (R)

When we plot x^2 versus H , the x-intercept of the line of best fit is an estimated value of R . We get the value of R to be about 0.050 m , with the uncertainty of 0.0115 m .

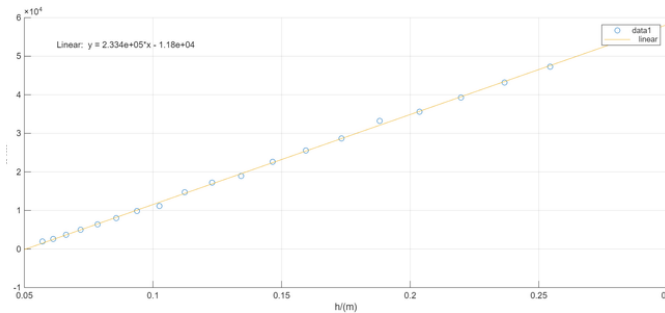


Figure 4: Determination of Residual Head (R). The square of the projectile range (x^2) is plotted against height (H). The x-intercept represents the theoretical height at which flow ceases ($H = R$), utilized here to quantify the energy loss due to the *vena contracta* effect.

5.3 Universal Dimensionless Curves

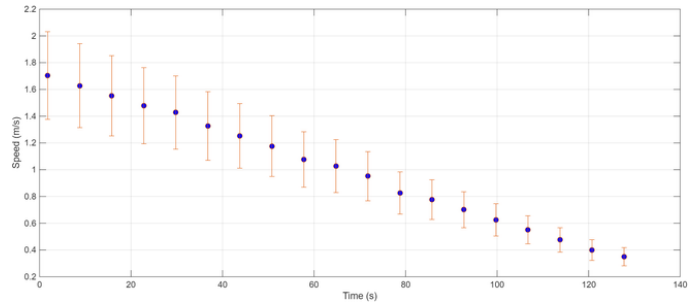


Figure 5: Discharge velocity (v) plotted against time (t). The observed linear decrease suggests a constant deceleration of the fluid surface level, consistent with the theoretical derivation for a cylindrical tank.

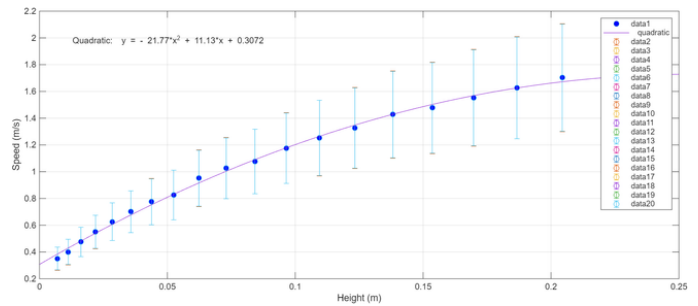


Figure 6: Discharge velocity (v) as a function of fluid height (H). The non-linear trend aligns with the theoretical prediction v is directly proportional to the square root of H , though deviations are observed at low heights due to surface tension dominance.

We have universalised Torricelli's Law. As expected, v and t exhibit a linear relationship, while h and t display a quadratic relationship.

6. Discussion

6.1 Deviation from Ideal Theory

The most significant finding of this experiment is the divergence of the discharge behaviour from the ideal Torricelli model. While the plot of v^2 versus H (Figure 3) exhibits the expected linear trend, the experimental gradient was calculated to be 14.3 m/s^2 . This is substantially lower than the theoretical value of $2g_{\text{approx}}$ which is 19.6 m/s^2 .

This discrepancy represents a "surprise" that contradicts the simplified textbook assumption of energy conservation in an ideal fluid. This loss of kinetic energy is not merely a measurement error but a physical necessity caused by two dominant factors:

1. Vena Contracta: As fluid streamlines converge at the sharp-edged pinhole, the effective cross-sectional area of the jet becomes smaller than the physical area of the orifice. This constriction reduces the mass flow rate below theoretical predictions.

1. Viscous Dissipation: Real fluids possess viscosity, generating internal friction that dissipates potential energy as heat rather than kinetic energy.

By introducing the residual head (R), we successfully quantified this energy "tax." The non-zero x-intercept in Figure 5 indicates that the fluid behaves as if it has **0.050 m less depth** than physically measured, effectively correcting the Bernoulli equation for real-world inefficiencies.

6.2 Universalisation of Flow Dynamics

A key objective of this study was to move beyond specific apparatus dimensions. By normalising the data into dimensionless quantities (v and t), we demonstrated that the flow decay law is scale-invariant. As shown in Figure 6, the experimental data points for dimensionless velocity overlap almost perfectly with the theoretical line $v = 1 - t$. This confirms that the linear decay of discharge velocity is a universal property of cylindrical reservoirs, valid regardless of the tank size or fluid density, provided the geometry remains consistent.

6.3 Surface Tension at Low Heights

A secondary deviation was observed as H approaches 0. The data scatter increased significantly in this region (visible in the tail of Figure 1). This marks a transition from the "inertial regime" (dominated by gravity) to the "capillary regime" (dominated by surface tension). At very low heights, the hydrostatic pressure is insufficient to overcome surface tension at the pinhole, causing the flow to switch from a continuous jet to varying drips. This non-linear behaviour validates the necessity of excluding the final millimetres of drainage from the regression analysis to preserve the accuracy of the hydrodynamic model.

7. Conclusion

The experiment successfully validated the relationship between discharge velocity and fluid height. By determining a residual head of $R = 0.050$ with an uncertainty of 0.0115 m , we successfully accounted for the vena contracta effect. Furthermore, the derivation and plotting of dimensionless quantities $v = 1-t$ provided a universal verification of Torricelli's law that is independent of specific container geometry.

8. References

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6. *PhysLogger Desktop Application Documentation*, PhysLab, LUMS. (Referenced regarding data acquisition setup and sampling rates).