Assignment 5

Solution

Question 1:-

a)

$$\vec{a} = a\hat{i}$$

$$\vec{b} = a\left[\cos 60^\circ \hat{i} + \sin 60^\circ \hat{j}\right]$$

$$= \frac{a}{2}\hat{i} + \frac{\sqrt{3}}{2}a\hat{j}$$

b)

$$\vec{a}^* = \frac{2\pi}{a}\frac{\vec{b}\times\hat{z}}{\vec{a}\cdot(\vec{b}\times\hat{z})} \quad ; \quad \vec{b}^* = \frac{2\pi}{a}\frac{\hat{z}\times\vec{a}}{\vec{a}\cdot(\vec{b}\times\hat{z})}$$

$$\vec{b}\times\hat{z} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{z} \\ a/2 & \sqrt{3}a/2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{\sqrt{3}}{2}a\hat{i} - \frac{a}{2}\hat{j}$$

$$\vec{a}\cdot(\vec{b}\times\hat{z}) = \frac{\sqrt{3}}{2}a^2$$

$$\hat{z}\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{z} \\ 0 & 0 & 1 \\ a & 0 & 0 \end{vmatrix} = a\hat{j}$$

$$\Rightarrow \vec{a}^* = 2\pi\left(\frac{2}{\sqrt{3}a^2}\right)\left[\frac{\sqrt{3}}{2}a\hat{i} - \frac{a}{2}\hat{j}\right] = \frac{2\pi}{\sqrt{3}a}\left(\sqrt{3}\hat{i} - \hat{j}\right)$$

$$\vec{b}^* = 2\pi\left(\frac{2}{\sqrt{3}\,a^2}\right)(a\hat{j}) = \frac{4\pi}{\sqrt{3}\,a}\hat{j}$$

To find that the Wigner-Seitz Cell is hexagonal:-

$$\vec{a}^* \cdot \vec{b}^* = \left(\frac{-2\pi}{\sqrt{3}\,a}\right)\left(\frac{4\pi}{\sqrt{3}\,a}\right) = -\frac{8\pi^2}{3a^2}$$

$$|\vec{a}^*| = \sqrt{\frac{4\pi^2}{a^2} + \frac{4\pi^2}{3a^2}} = \frac{2\pi}{a}\sqrt{\frac{4}{3}}$$

$$= \frac{4\pi}{\sqrt{3}\,a}$$

$$|\vec{b}^*| = \frac{4\pi}{\sqrt{3}\,a}$$

$$\cos\theta = \frac{\vec{a}^* \cdot \vec{b}^*}{|\vec{a}^*||\vec{b}^*|} = \left(-\frac{8\pi^2}{3a^2}\right) \Big/ \left(\frac{16\pi^2}{3a^2}\right)$$

$$= -\frac{1}{2}$$

$$\Rightarrow \theta = 120^\circ$$

or $\theta = 180^\circ - 120^\circ = 60^\circ$

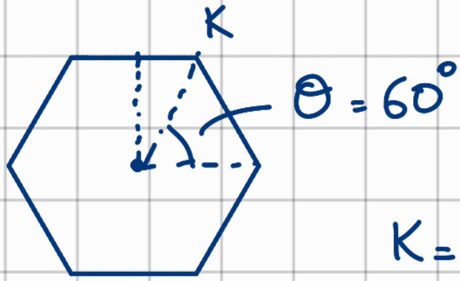
This makes Wigner-Seitz cell around $\Gamma = (0,0)$ a regular hexagon.

Coordinates of :-

- M : midpoint of upper horizontal edge of hexagon

$$\Rightarrow \quad M = \left(0, \frac{4\pi}{\sqrt{3}a}\left(\frac{1}{2}\right)\right) = \left(0, \frac{2\pi}{\sqrt{3}a}\right)$$

- K :



$$K = \frac{4\pi}{\sqrt{3}a}\left(\cos 60°, \sin 60°\right)$$
$$= \left(\frac{2\pi}{\sqrt{3}a}, \frac{2\pi}{a}\right)$$

c) $$\Phi^{1,x}_{0,x} = -\alpha\,(\hat{\omega}_1 \cdot \hat{x})(\hat{\omega}_1 \cdot \hat{x}) = -\alpha\,(1)(1)$$
$$= -\alpha$$
where $\hat{\omega}_1 = (1,0)$

Similarly,
$$\Phi^{1,x}_{0,y} = \Phi^{1,y}_{0,x} = \Phi^{1,y}_{0,y} = 0$$

With $\hat{\omega}_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$:-

$$\Phi^{2,x}_{0,x} = -\alpha\,(\hat{\omega}_2 \cdot \hat{x})(\hat{\omega}_2 \cdot \hat{x})$$
$$= \frac{-\alpha}{4}$$

$$\Phi^{2,x}_{0,y} = \Phi^{2,y}_{0,x} = -\alpha\,(\hat{\omega}_2 \cdot \hat{x})(\hat{\omega}_2 \cdot \hat{y})$$
$$= -\frac{\sqrt{3}}{4}\alpha$$

$$\Phi^{2,y}_{0,y} = -\alpha\left(\frac{\sqrt{3}}{2}\right)^2 = -\frac{3}{4}\alpha$$

With $\hat{\omega}_3 = \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$:-

$$\Phi^{3,x}_{0,x} = -\alpha\left(\hat{\omega}_3 \cdot \hat{x}\right)\left(\hat{\omega}_3 \cdot \hat{x}\right)$$
$$= \frac{-\alpha}{4}$$

$$\Phi^{3,x}_{0,y} = \Phi^{3,y}_{0,x} = -\alpha\left(\hat{\omega}_3 \cdot \hat{x}\right)\left(\hat{\omega}_3 \cdot \hat{y}\right)$$
$$= +\frac{\sqrt{3}}{4}\alpha$$

$$\Phi^{3,y}_{0,y} = -\alpha\left(\frac{\sqrt{3}}{2}\right)^2 = -\frac{3}{4}\alpha$$

With $\hat{\omega}_4 = \left(-1, 0\right)$:-

$$\Phi^{4,x}_{0,x} = -\alpha\left(\hat{\omega}_4 \cdot \hat{x}\right)\left(\hat{\omega}_4 \cdot \hat{x}\right)$$
$$= -\alpha$$

$$\Phi^{4,x}_{0,y} = \Phi^{4,y}_{0,x} = \Phi^{4,y}_{0,y} = 0$$

With $\hat{\omega}_5 = \left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right)$:-

$$\Phi^{5,x}_{0,x} = -\alpha\left(\hat{\omega}_5 \cdot \hat{x}\right)\left(\hat{\omega}_5 \cdot \hat{x}\right)$$
$$= \frac{-\alpha}{4}$$

$$\Phi^{5,x}_{0,y} = \Phi^{5,y}_{0,x} = -\alpha\left(\hat{\omega}_5 \cdot \hat{x}\right)\left(\hat{\omega}_5 \cdot \hat{y}\right)$$
$$= -\frac{\sqrt{3}}{4}\alpha$$

$$\Phi^{5,y}_{0,y} = -\alpha\left(-\frac{\sqrt{3}}{2}\right)^2 = -\frac{3}{4}\alpha$$

With $\hat{\omega}_6 = \left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$:-

$$\Phi^{6,x}_{0,x} = -\alpha\left(\hat{\omega}_6 \cdot \hat{x}\right)\left(\hat{\omega}_6 \cdot \hat{x}\right) = \frac{-\alpha}{4}$$

$$\Phi^{6,x}_{0,y} = \Phi^{6,y}_{0,x} = -\alpha\left(\hat{\omega}_6 \cdot \hat{x}\right)\left(\hat{\omega}_6 \cdot \hat{y}\right) = +\frac{\sqrt{3}}{4}\alpha$$

$$\Phi^{6,y}_{0,y} = -\alpha\left(\frac{\sqrt{3}}{2}\right)^2 = -\frac{3}{4}\alpha$$

d) $$D^{m,j}_{n,i} = \frac{1}{M}\sum_m \Phi^{m,j}_{n,i}\, e^{i\vec{k}\cdot(\vec{r}_m - \vec{r}_n)}$$

$$D^{m,x}_{0,x} = \frac{1}{M}\sum_m \Phi^{m,x}_{0,x}\, e^{i\vec{k}\cdot(\vec{r}_m - \vec{r}_0)}$$

$$= \frac{1}{M}\Big[-\alpha e^{ik_x a} - \alpha e^{-ik_x a} - \frac{\alpha}{4} e^{ik_x a/2} e^{ik_y\sqrt{3}a/2} + - \frac{\alpha}{4} e^{-ik_x a/2} e^{ik_y\sqrt{3}a/2} - \frac{\alpha}{4} e^{-ik_x a/2} e^{-ik_y\sqrt{3}a/2} + - \frac{\alpha}{4} e^{ik_x a/2} e^{-ik_y\sqrt{3}a/2}\Big]$$

$$= \frac{-\alpha}{4M}\left[8\cos(k_x a) + 2e^{ik_x a/2}\cos\left(\frac{\sqrt{3}k_y a}{2}\right) + + 2e^{-ik_x a/2}\cos\left(\frac{\sqrt{3}k_y a}{2}\right)\right]$$

$$= \frac{-\alpha}{M}\left[2\cos(k_x a) + \cos\left(\frac{k_x a}{2}\right)\cos\left(\frac{\sqrt{3}k_y a}{2}\right)\right]$$

$$D_{o,y}^{m,x} = \frac{1}{M}\sum_m \Phi_{o,y}^{m,x} e^{i\vec{k}\cdot(\vec{r}_m - \vec{r}_o)}$$

$$= \frac{1}{M}\left[-\frac{\sqrt{3}}{4}\alpha e^{ik_x a/2} e^{ik_y\sqrt{3}a/2} + \cancel{\frac{\sqrt{3}}{4}\alpha e^{-ik_x a/2} e^{ik_y\sqrt{3}a/2}}\right.$$

$$\left. \cancel{-\frac{\sqrt{3}}{4}\alpha e^{-ik_x a/2} e^{-ik_y\sqrt{3}a/2}} + \frac{\sqrt{3}}{4}\alpha e^{ik_x a/2} e^{-ik_y\sqrt{3}a/2}\right]$$

$$= \frac{\sqrt{3}}{M}\alpha \sin\left(\frac{k_x a}{2}\right)\sin\left(\frac{\sqrt{3}k_y a}{2}\right)$$

$$\therefore D_{o,x}^{m,y} = D_{o,y}^{m,x} = \frac{\sqrt{3}}{M}\alpha \sin\left(\frac{k_x a}{2}\right)\sin\left(\frac{\sqrt{3}k_y a}{2}\right)$$

$$D_{o,y}^{m,y} = -\frac{3}{4M}\alpha\left[e^{ik_x a/2} e^{ik_y\sqrt{3}a/2} + e^{-ik_x a/2} e^{ik_y\sqrt{3}a/2} + \right.$$

$$\left. e^{-ik_x a/2} e^{-ik_y\sqrt{3}a/2} + e^{ik_x a/2} e^{-ik_y\sqrt{3}a/2}\right]$$

$$= \frac{-3}{M}\alpha \cos\left(\frac{k_x a}{2}\right)\cos\left(\frac{\sqrt{3}k_y a}{2}\right)$$

$$|D - \omega^2 \mathbb{1}| = 0$$

$$\Rightarrow \left[\frac{-\alpha}{M}\left[2\cos(k_x a) + \cos\left(\frac{k_x a}{2}\right)\cos\left(\frac{\sqrt{3}k_y a}{2}\right)\right] - \omega^2\right] \times$$

$$\left[\frac{-3}{M}\alpha \cos\left(\frac{k_x a}{2}\right)\cos\left(\frac{\sqrt{3}k_y a}{2}\right) - \omega^2\right] +$$

$$-\frac{3}{M^2}\alpha^2 \sin^2\left(\frac{k_x a}{2}\right)\sin^2\left(\frac{\sqrt{3}k_y a}{2}\right) = 0$$

```
In[•]:= Q = DiagonalMatrix[{-ω^2 + (2 + Cos[ky*Sqrt[3]/2] - 3), -ω^2 + 3*(Cos[ky*Sqrt[3]/2] - 1)}]
```

Out[•]= $\left\{\left\{-1-\omega^2+\text{Cos}\left[\frac{\sqrt{3}\,\text{ky}}{2}\right], 0\right\}, \left\{0, -\omega^2+3\left(-1+\text{Cos}\left[\frac{\sqrt{3}\,\text{ky}}{2}\right]\right)\right\}\right\}$

```
In[•]:= Solve[Simplify[Det[Q]] == 0, ω]
```

Out[•]= $\left\{\left\{\omega\to-\sqrt{-1+\text{Cos}\left[\frac{\sqrt{3}\,\text{ky}}{2}\right]}\right\}, \left\{\omega\to\sqrt{-1+\text{Cos}\left[\frac{\sqrt{3}\,\text{ky}}{2}\right]}\right\}, \left\{\omega\to-\sqrt{3}\sqrt{-1+\text{Cos}\left[\frac{\sqrt{3}\,\text{ky}}{2}\right]}\right\}, \left\{\omega\to\sqrt{3}\sqrt{-1+\text{Cos}\left[\frac{\sqrt{3}\,\text{ky}}{2}\right]}\right\}\right\}$

```
Q = DiagonalMatrix[{-ω^2 + (2 + Cos[ky*Sqrt[3]/2] - 3), -ω^2 + 3*(Cos[ky*Sqrt[3]/2] - 1)}]
```

```
In[•]:= A = {{-ω^2 + α/M*(3 - 2*Cos[kx*a] - Cos[kx*a/2]*Cos[ky*Sqrt[3]*a/2]), Sqrt[3]*α/M*Sin[kx*a/2]*Sin[ky*Sqrt[3]*a/2]},
  {Sqrt[3]*α/M*Sin[kx*a/2]*Sin[ky*Sqrt[3]*a/2], -ω^2 + 3*α/M*(1 - Cos[kx*a/2]*Cos[ky*Sqrt[3]*a/2])}}
```

Out[•]= $\left\{\left\{-\omega^2+\frac{\alpha\left(3-2\,\text{Cos}[a\,\text{kx}]-\text{Cos}\left[\frac{a\,\text{kx}}{2}\right]\text{Cos}\left[\frac{1}{2}\sqrt{3}\,a\,\text{ky}\right]\right)}{M}, \frac{\sqrt{3}\,\alpha\,\text{Sin}\left[\frac{a\,\text{kx}}{2}\right]\text{Sin}\left[\frac{1}{2}\sqrt{3}\,a\,\text{ky}\right]}{M}\right\}, \left\{\frac{\sqrt{3}\,\alpha\,\text{Sin}\left[\frac{a\,\text{kx}}{2}\right]\text{Sin}\left[\frac{1}{2}\sqrt{3}\,a\,\text{ky}\right]}{M}, -\omega^2+\frac{3\alpha\left(1-\text{Cos}\left[\frac{a\,\text{kx}}{2}\right]\text{Cos}\left[\frac{1}{2}\sqrt{3}\,a\,\text{ky}\right]\right)}{M}\right\}\right\}$

e)

From Γ to M

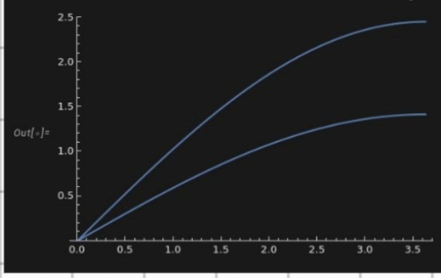
```
In[•]:= FullSimplify[Solve[FullSimplify[Det[A /. kx → 0]] == 0, ω]]
```

Out[•]= $\left\{\left\{\omega\to-\sqrt{\frac{4M\alpha\,\text{Sin}\left[\frac{1}{4}\sqrt{3}\,a\,\text{ky}\right]^2-2\sqrt{M^2\alpha^2\,\text{Sin}\left[\frac{1}{4}\sqrt{3}\,a\,\text{ky}\right]^4}}{M^2}}\right\}, \left\{\omega\to\sqrt{\frac{4M\alpha\,\text{Sin}\left[\frac{1}{4}\sqrt{3}\,a\,\text{ky}\right]^2-2\sqrt{M^2\alpha^2\,\text{Sin}\left[\frac{1}{4}\sqrt{3}\,a\,\text{ky}\right]^4}}{M^2}}\right\}, \left\{\omega\to-\sqrt{2}\sqrt{\frac{2M\alpha\,\text{Sin}\left[\frac{1}{4}\sqrt{3}\,a\,\text{ky}\right]^2+\sqrt{M^2\alpha^2\,\text{Sin}\left[\frac{1}{4}\sqrt{3}\,a\,\text{ky}\right]^4}}{M^2}}\right\}, \left\{\omega\to\sqrt{2}\sqrt{\frac{2M\alpha\,\text{Sin}\left[\frac{1}{4}\sqrt{3}\,a\,\text{ky}\right]^2+\sqrt{M^2\alpha^2\,\text{Sin}\left[\frac{1}{4}\sqrt{3}\,a\,\text{ky}\right]^4}}{M^2}}\right\}\right\}$

In[•]:= q1 = Plot $\left[\text{FullSimplify}\left[\left\{\sqrt{\frac{4M\alpha\,\text{Sin}\left[\frac{1}{4}\sqrt{3}\,a\,\text{ky}\right]^2-2\sqrt{M^2\alpha^2\,\text{Sin}\left[\frac{1}{4}\sqrt{3}\,a\,\text{ky}\right]^4}}{M^2}}, \sqrt{2}\sqrt{\frac{2M\alpha\,\text{Sin}\left[\frac{1}{4}\sqrt{3}\,a\,\text{ky}\right]^2+\sqrt{M^2\alpha^2\,\text{Sin}\left[\frac{1}{4}\sqrt{3}\,a\,\text{ky}\right]^4}}{M^2}}\right\}\right] \;/.\; \{\alpha\to1, a\to1, M\to1\},\right.$

```
{ky, 0, 2*Pi/Sqrt[3]}, PlotRange → {0, 2.5}]
```

Out[•]=



From M to K

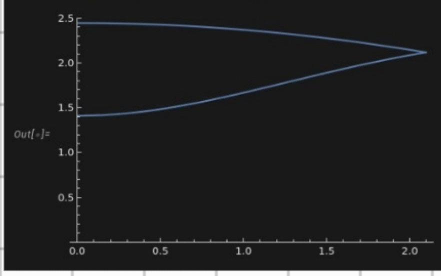
```
In[•]:= FullSimplify[Solve[FullSimplify[Det[A /. ky → 2*Pi/(a*Sqrt[3])]] == 0, ω]]
```

Out[•]= $\left\{\left\{\omega\to-\sqrt{\frac{-2\sqrt{M^2\alpha^2\,\text{Cos}\left[\frac{a\,\text{kx}}{4}\right]^4\left(1-2\,\text{Cos}\left[\frac{a\,\text{kx}}{2}\right]\right)^2}+M\alpha\left(3+2\,\text{Cos}\left[\frac{a\,\text{kx}}{2}\right]-\text{Cos}[a\,\text{kx}]\right)}{M^2}}\right\}, \left\{\omega\to\sqrt{\frac{-2\sqrt{M^2\alpha^2\,\text{Cos}\left[\frac{a\,\text{kx}}{4}\right]^4\left(1-2\,\text{Cos}\left[\frac{a\,\text{kx}}{2}\right]\right)^2}+M\alpha\left(3+2\,\text{Cos}\left[\frac{a\,\text{kx}}{2}\right]-\text{Cos}[a\,\text{kx}]\right)}{M^2}}\right\}, \left\{\omega\to-\sqrt{\frac{2\sqrt{M^2\alpha^2\,\text{Cos}\left[\frac{a\,\text{kx}}{4}\right]^4\left(1-2\,\text{Cos}\left[\frac{a\,\text{kx}}{2}\right]\right)^2}+M\alpha\left(3+2\,\text{Cos}\left[\frac{a\,\text{kx}}{2}\right]-\text{Cos}[a\,\text{kx}]\right)}{M^2}}\right\}, \left\{\omega\to\sqrt{\frac{2\sqrt{M^2\alpha^2\,\text{Cos}\left[\frac{a\,\text{kx}}{4}\right]^4\left(1-2\,\text{Cos}\left[\frac{a\,\text{kx}}{2}\right]\right)^2}+M\alpha\left(3+2\,\text{Cos}\left[\frac{a\,\text{kx}}{2}\right]-\text{Cos}[a\,\text{kx}]\right)}{M^2}}\right\}\right\}$

In[•]:= q2 = Plot $\left[\text{FullSimplify}\left[\left\{\sqrt{\frac{2\sqrt{M^2\alpha^2\,\text{Cos}\left[\frac{a\,\text{kx}}{4}\right]^4\left(1-2\,\text{Cos}\left[\frac{a\,\text{kx}}{2}\right]\right)^2}+M\alpha\left(3+2\,\text{Cos}\left[\frac{a\,\text{kx}}{2}\right]-\text{Cos}[a\,\text{kx}]\right)}{M^2}}, \sqrt{\frac{-2\sqrt{M^2\alpha^2\,\text{Cos}\left[\frac{a\,\text{kx}}{4}\right]^4\left(1-2\,\text{Cos}\left[\frac{a\,\text{kx}}{2}\right]\right)^2}+M\alpha\left(3+2\,\text{Cos}\left[\frac{a\,\text{kx}}{2}\right]-\text{Cos}[a\,\text{kx}]\right)}{M^2}}\right\}\right] \;/.\right.$

```
{α → 1, a → 1, M → 1}, {kx, 0, 2*Pi/3}, PlotRange → {0, 2.5}]
```

Out[•]=



From K to Γ

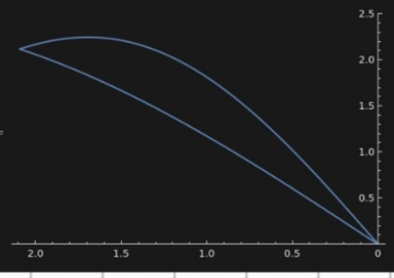
In[•]:= `Solve[Det[A /. ky → Sqrt[3] * kx] == 0, ω]`

Out[•]= $\left\{\left\{\omega \to -\frac{\sqrt{3}\,\sqrt{\alpha - \alpha\,\mathrm{Cos}[a\,kx]}}{\sqrt{M}}\right\}, \left\{\omega \to \frac{\sqrt{3}\,\sqrt{\alpha - \alpha\,\mathrm{Cos}[a\,kx]}}{\sqrt{M}}\right\}, \left\{\omega \to -\frac{\sqrt{3\,\alpha - \alpha\,\mathrm{Cos}[a\,kx] - 2\,\alpha\,\mathrm{Cos}[2\,a\,kx]}}{\sqrt{M}}\right\}, \left\{\omega \to \frac{\sqrt{3\,\alpha - \alpha\,\mathrm{Cos}[a\,kx] - 2\,\alpha\,\mathrm{Cos}[2\,a\,kx]}}{\sqrt{M}}\right\}\right\}$

In[•]:= q3 = Plot$\left[\mathrm{FullSimplify}\left[\left\{\frac{\sqrt{3}\,\sqrt{\alpha - \alpha\,\mathrm{Cos}[a\,kx]}}{\sqrt{M}}, \frac{\sqrt{3\,\alpha - \alpha\,\mathrm{Cos}[a\,kx] - 2\,\alpha\,\mathrm{Cos}[2\,a\,kx]}}{\sqrt{M}}\right\}\right]\right.$ `/. {α → 1, a → 1, M → 1}, {kx, 2 * Pi / 3, 0}, ScalingFunctions → {"Reverse", Identity},`

`PlotRange → {0, 2.5}]`

Out[•]=



Question 2:-

a) $$s_n = u_n e^{i(kna-\omega t)} = u_n e^{i(Kna-\omega t)} e^{-\eta na}$$

For a monoatomic 1D lattice,

$$\ddot{s}_n = \frac{\alpha}{M}\left(s_{n+1} + s_{n-1} - 2s_n\right)$$

$$-\omega^2 s_n = \frac{\alpha}{M}\left(s_n e^{i(K+i\eta)a} + s_n e^{-i(K+i\eta)a} - 2s_n\right)$$

For arbitrary s_n:

$$-M\omega^2 = \alpha\left(e^{i(K+i\eta)a} + e^{-i(K+i\eta)a} - 2\right)$$

$$\cosh\beta = \frac{e^{\beta}+e^{-\beta}}{2}$$

$$\rightarrow \cosh\left[i(K+i\eta)\right] = \cos(K+i\eta) = \cos(K)\cos(i\eta) - \sin(K)\sin(i\eta)$$
$$= \cos(K)\cosh(\eta) - i\sin(K)\sinh(\eta)$$

as $\cosh(i\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2} = \cos\theta$

$$\cos(i\eta) = \frac{e^{+i(i\eta)} + e^{-i(i\eta)}}{2} = \frac{e^{-\eta}+e^{\eta}}{2} = \cosh(\eta)$$

$$\sin(i\eta) = \frac{e^{i(i\eta)} - e^{-i(i\eta)}}{2} = \frac{e^{-\eta}-e^{\eta}}{2i} = i\sinh(\eta)$$

$$\Rightarrow D = \frac{2\alpha}{M}\left[1 - \cos(K)\cosh(\eta) + i\sin(K)\sinh(\eta)\right]$$

$$\omega = \sqrt{\frac{2\alpha}{M}}\sqrt{1-\cos(K)\cosh(\eta)+i\sin(K)\sinh(\eta)}$$

is complex

If $\eta << K$ (η is small),

$$\cosh(\eta) = \frac{e^{\eta}-e^{\eta}}{2} \approx \frac{1}{2}(1+\eta+1-\eta) \approx 1$$

$$\sinh(\eta) \approx \frac{1}{2}(1+\eta-1+\eta) \approx \eta$$

$$\therefore \quad \omega = \sqrt{\frac{2\alpha}{M}}\sqrt{1-\cos(K)+i\eta\sin(K)}$$

$\Rightarrow$ K small $\longrightarrow$ acoustic, near $k=0$

$$\omega = \sqrt{\frac{2\alpha}{M}}\sqrt{1-\left(1-\frac{K^2}{2}\right)+iK\eta}$$

$$\omega = \sqrt{\frac{2\alpha}{M}}\left(\frac{K^2}{2}+iK\eta\right)^{1/2} = \sqrt{\frac{\alpha K}{M}}\left[1+iK\eta\frac{2}{K^2}\right]^{1/2}$$

b) as $K >> \eta$,

$$\omega = \sqrt{\frac{\alpha K}{M}}\left(1+\frac{i\eta}{K}\right)$$

$$Re(\omega) = \sqrt{\frac{\alpha K}{M}} \quad ; \quad Im(\omega) = \sqrt{\frac{\alpha K}{M}}\frac{\eta}{K}$$