Assignment 6

Solution

Question 1:-

a) $E(k) = \frac{\hbar^2 k^2}{2m} \Rightarrow dE = \frac{\hbar^2 k}{m} dk \Rightarrow \frac{dk}{dE} = \frac{m}{\hbar^2 k}$

Because of $k_\pm$ and being confined to a 1D chain:-

$$dN = 2 \frac{dk}{2\pi/L} = \frac{L}{\pi} dk$$

$$\Rightarrow D(E) = \frac{dN}{dE}$$

$$= \frac{L}{\pi} \frac{m}{\hbar^2 k}$$

$$= \frac{L}{\pi} \frac{m}{\hbar} \frac{1}{\sqrt{2mE}}$$

$$D(E) \propto E^{-1/2}$$

b) As we are now confined to a 2D plane:

$$dN = 2 \frac{1}{(2\pi/L)^2} d^2k = 2 \frac{L^2}{4\pi^2} (2\pi) k\,dk$$

$$= \frac{L^2}{\pi} \frac{\sqrt{2mE}}{\hbar} \left(\frac{m}{\hbar^2 k} \right)$$

$$= \frac{L^2}{\pi} \frac{\sqrt{2\cancel{m}E}}{\cancel{\hbar}} \left(\frac{m}{\hbar^2} \right) \frac{\cancel{\hbar}}{\sqrt{\cancel{2mE}}}$$

$$D(E) = \frac{L^2 m}{\pi \hbar^2}$$

$D(E)$ is independent of E

Question 2:-

When an external magnetic field $\vec{B}$ is applied, the energies of two spin-states become:

$$E_\uparrow = E - \mu_B B \quad ; \quad E_\downarrow = E + \mu_B B$$

where $\mu_B \equiv \frac{e\hbar}{2m}$ (Bohr magneton)

As $M = \mu_B(n_\uparrow - n_\downarrow)$, and for weak fields,

$$n_\uparrow \approx \int_0^{E_F} dE \; D(E - \mu_B B)$$

$$n_\downarrow \approx \int_0^{E_F} dE \; D(E + \mu_B B)$$

Taylor expanding :

$$D(E - \mu_B B) = D(E) + (-\mu_B B)\frac{\partial D}{\partial E} + \dots$$

$$D(E + \mu_B B) = D(E) + (\mu_B B)\frac{\partial D}{\partial E} + \dots$$

$$\Rightarrow \quad n_\uparrow - n_\downarrow \approx \int_0^{E_F} -2\mu_B B \frac{\partial D}{\partial E} \, dE + \dots$$

$$= -2\mu_B B \, D(E_F)$$

$$\Rightarrow M = -2\mu_B^2 B D(E_F)$$

$$\Rightarrow \chi = \frac{M\mu_B}{B} = -2\mu_B^2\mu_0 D(E_F)$$

For a 3D free electron gas,

$$D(E_F) = \frac{3n}{2E_F}$$

$$\Rightarrow \chi = -\frac{3n\mu_0\mu_B^2}{E_F}$$

(makes no reference to temperature T, hence it is independent of T)

Question 3 :-

From Q.1-b), for 2D :-

$$D(E) = \frac{L^2 m}{\pi \hbar^2}$$

$$U = \int_0^{E_F} E\, D(E)\, f_{FD}$$

$$= \frac{L^2 m}{\pi \hbar^2} \int_0^{E_F} \frac{E}{e^{(E-\mu)/k_B T} + 1}\, dE$$

Use Sommerfeld expansion

$$I = \int dE\, f(E)\, \Gamma'(E)$$

$$= \Gamma(\mu) + \frac{\pi^2}{6} (k_B T)^2\, \Gamma''(\mu)$$

where $\Gamma'(E) = E D(E)$

$$= \frac{L^2 m}{\pi \hbar^2} E$$

$$\Rightarrow \Gamma(E) = \frac{L^2 m}{2\pi \hbar^2} E^2$$

$$\Rightarrow \Gamma''(E) = \frac{L^2 m}{\pi \hbar^2}$$

$$\Rightarrow I = \Gamma(\mu) + \frac{\pi^2}{6} (k_B T)^2\, \Gamma''(\mu)$$

$$I = \frac{L^2 m}{2\pi\hbar^2}\mu^2 + \frac{\pi^2}{6}(k_B T)^2 \frac{L^2 m}{\pi\hbar^2}$$

$$\Rightarrow \quad U = \frac{L^2 m}{2\pi\hbar^2}\mu^2 + \frac{\pi^2}{6}(k_B T)^2 \frac{L^2 m}{\pi\hbar^2}$$

$$\therefore \quad C_v = \left.\frac{\partial U}{\partial T}\right|_v = \frac{\pi^2}{3}\frac{L^2 m}{\pi\hbar^2} k_B^2 T$$

$C_v \propto T$ in low temp. regime

Question 4:-

$$N = 2\left(\frac{4}{3}\pi k_F^3\right)\left(\frac{1}{(2\pi/L)^3}\right) \quad ; \quad V = L^3$$

$$= \frac{V k_F^3}{3\pi^2}$$

$$\Rightarrow \quad n = \frac{N}{V} = \frac{k_F^3}{3\pi^2}$$

$$\Rightarrow \quad k_F = (3\pi^2 n)^{1/3}$$

As $$E = k_B T_F = \frac{\hbar^2 k_F^2}{2m}$$

$$\therefore T_F = \frac{\hbar^2 (3\pi^2 n)^{2/3}}{2m k_B} \quad ; \quad n = \frac{\rho}{m} \text{ (in question)}$$

$$T_F = \frac{\hbar^2 (3\pi^2 \rho/m)^{2/3}}{2m k_B}$$

a) $$\rho = 81\,kg\,m^{-3} \quad ; \quad m_{^3He} = 3u = 3(1.66\times10^{-27})$$

$$T_F = \frac{(1.055\times10^{-34})^2\left[3\pi^2(81/(3)(1.66\times10^{-27})\right]^{2/3}}{2(3)(1.66\times10^{-27})(1.38\times10^{-23})}$$

$$\approx 5K$$

b) $$\rho = 10^{17}\,kg\,m^{-3} \quad ; \quad m_n = 1.675\times10^{-27}\,kg$$

$$T_F = \frac{(1.055\times10^{-34})^2\left[3\pi^2(81/(1.675\times10^{-27})\right]^{2/3}}{2(1.675\times10^{-27})(1.38\times10^{-23})} \approx 10^{11}K$$

Question 5:-

a)
$$dN = \frac{1}{(2\pi/L)^3} d^3k = \frac{V}{(2\pi)^3} 4\pi k^2 dk$$

$$E(k) = \frac{\hbar^2 k^2}{2m}, \quad dk = \frac{m}{\hbar^2 k} dE$$

$$dN = g(E)\,dE = \frac{2(4\pi)V}{8\pi^3} k^2 dk$$

$$= \frac{2(4\pi)V}{8\pi^3} k^2 \left(\frac{m}{\hbar^2 k}\right) dE$$

$$= \frac{2(4\pi)V}{8\pi^3} \left(\frac{2mE}{\hbar^2}\right)^{1/2} \left(\frac{m}{\hbar^2}\right) dE$$

$$g(E)\,dE = \frac{V}{2\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2} dE$$

$$\therefore \; g(E) = \frac{V}{2\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2} dE$$

in 3D

$$\Rightarrow \; U = \int_0^{E_F} E\, g(E)\, dE \quad \text{at } 0K$$

$$= \frac{V}{2\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^{E_F} E^{3/2} dE$$

$$= \frac{V}{\cancel{2}\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{\cancel{2}E_F^{5/2}}{5}$$

$$U = \frac{V}{5\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_F^{5/2}$$

Since total no. of electrons are

$$N = \int_0^{E_F} g(E)\, dE$$

$$= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^{E_F} E^{1/2}\, dE$$

$$N = \frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_F^{3/2}$$

$$\Rightarrow E_F = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3}$$

Therefore,

$$U = \frac{3}{5} N \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3}$$

b)

$$P = -\frac{\partial U}{\partial V}\bigg|_N$$

$$= \frac{\cancel{3}}{5} N \frac{\hbar^2}{\cancel{2}m}\left(3\pi N\right)^{2/3}\left(-\frac{\cancel{2}}{\cancel{3}}\right) V^{-1/3}$$

$$P = -\frac{N}{5m}\left(\frac{3\pi N}{V^{1/2}}\right)^{2/3}$$

c)

$$U = \frac{V}{5\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_F^{5/2} \; ; \quad N = \frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_F^{3/2}$$

$$\Rightarrow \quad U = \frac{3}{5}\frac{N}{V} E_F = \frac{3}{5} n E_F$$

Since $U \propto V^{-2/3}$

$$\left(\frac{1}{2}V\right)^{-2/3} = 4^{1/3} V^{-2/3} \propto U_s$$

$$\Delta U = U_s - U$$
$$= \left(4^{1/3} - 1\right) U = 0.587 U$$

For 2kg of ^{239}Pu,

$$N = \frac{2}{0.239} N_A \approx 5.04 \times 10^{24}$$

$$V = \frac{2}{\rho} = \frac{2}{1.9 \times 10^{4}} \approx 1.05 \times 10^{-4}\ m^3$$

$$\Rightarrow n = \frac{5.04 \times 10^{24}}{1.05 \times 10^{-4}} = 4.8 \times 10^{28}\ m^{-3}$$

$$\Rightarrow E_F = \frac{\hbar^2}{2m}\left(3\pi^2 (4.8 \times 10^{28})\right)^{2/3}$$
$$\approx 7.72 \times 10^{-19} = 4.83\,eV$$

$$\therefore\ U = \frac{3}{5}(4.8 \times 10^{28})(4.83\,eV)$$
$$U = 1.39 \times 10^{29}\ eV$$
$$U = 2.22 \times 10^{10}\ J$$

$$\therefore\ \Delta U = 0.587\,(2.22 \times 10^{10})$$
$$= 1.303 \times 10^{10}\ J$$

Required mass

$$M = \frac{\Delta U}{4M} = \frac{1.303 \times 10^{10}}{4 \times 10^{6}}$$

$$= 3.26 \times 10^{3}\ \text{kg}$$

d) Show:

$$U_{grav} = -\frac{3}{5}\frac{GM^2}{R}$$

The enclosed mass $M(r)$ of a brown dwarf :-

$$M(r) = \frac{4}{3}\pi r^3 \rho \qquad \text{where } \rho = \frac{M}{\frac{4}{3}\pi^2 R^3}$$

Total Mass

Total Volume

$$\Rightarrow M(r) = M\frac{r^3}{R^3}$$

Therefore, $dM = 4\pi r^2 \rho \, dr$.

Gravitational potential energy dU to bring dM from infinity to this shell:

$$dU = -\frac{GM(r)dm}{r}$$

$$dU = -\frac{GMr^3}{R^3 r}(4\pi r^2 \rho)\,dr$$

$$= -\frac{\cancel{4\pi} GM}{R^3 r}\left(\frac{3M}{\cancel{4\pi} R^3}\right) r^5 dr$$

$$dU = -\frac{3GM^2}{R^6} r^4 dr$$

$$U = -\frac{3GM^2}{R^6}\int_0^R r^4 dr$$

$$\therefore U = -\frac{3}{5}\frac{GM^2}{R}$$

$$U_e = \frac{3}{5} N E_F = \frac{3}{5} N \frac{\hbar^2}{2m_e}\left(3\pi^2 \frac{N}{V}\right)^{2/3}$$

$$= \frac{3\hbar^2}{10 m_e}(3\pi^2)^{2/3}\left(\frac{N^5}{V^2}\right)^{1/3}$$

$$= \frac{3\hbar^2}{10 m_e}(3\pi^2)^{2/3}\left(\frac{3}{4\pi^2}\right)^{5/3}\left(\frac{N^5}{R^{2(3)}}\right)^{1/3}$$

$$U_e = \frac{3\hbar^2}{10\pi^2 m_e}\left(\frac{3^7 N^5}{4^5}\right)^{1/3} R^{-2}$$

$$V = \frac{4\pi}{3}R^3$$

$$R = \left(\frac{3V}{4\pi}\right)^{1/3}$$

$$\Rightarrow U = U_e - U_{grav}$$

$$= \frac{3\hbar^2}{10\pi^2 m_e}\left(\frac{3^7 N^5}{4^5}\right)^{1/3}\frac{N}{R^2} + \frac{3}{5}\frac{GM^2}{R}$$

$$= \frac{3\hbar^2}{10 m_e}(3\pi^2)^{2/3}\frac{N^{5/3}}{V^{2/3}} + \frac{3}{5}GM^2\left(\frac{4\pi}{3V}\right)^{1/3}$$

$$\Rightarrow \frac{\partial U}{\partial V} = 0 = -\underbrace{\frac{2}{3}\frac{3\hbar^2(3\pi^2)^{2/3}N^{5/3}}{10 m_e}}_{\equiv A}\frac{1}{V^{5/3}} + \underbrace{\frac{(36\pi)^{1/3}G}{5}}_{\equiv B}\frac{M^2}{V^{4/3}}$$

$$-AV^{-5/3} + BM^2V^{-4/3} = 0 \Rightarrow AR^{-5} = BM^2R^{-4}$$

$$\frac{A}{B} = M^2R$$

$$M \propto 1/\sqrt{R}$$