Assignment 7

Density of States, Periodic Functions
Please no AI-generated solutions. Would be your loss!

Question 1

(a) Show that the density of states at Fermi energy is (in $3D$):

$$D(E_F) = \frac{3N}{2E_F}.$$

(b) Obtain a similar expression for $D(E_F)$ in $2D$.

Question 2

For Cu (monovalent $Z = 1$, $n = 8.45 \times 10^{28}$ atoms / m^3, $\sigma = 5.9 \times 10^7$ $(\Omega\ \text{m})^{-1}$ at 300 K), calculate:

(a) v_F at 300,

(b) v_d in a field of 10 V/m,

(c) mean free path ℓ.

Question 3

(a) Derive the predicted heat capacity C_V for electrons in a metal and show that the overall heat capacity (from electrons and phonons) at low temperatures is:

$$C_V = aT + bT^3.$$

(b) For potassium metal K, $a = 2.08$ mJ K^{-2} and $b = 2.6$ mJ mol^{-1}K^{-4}. Make an estimate for the Fermi energy of K.

Question 4

Show that (using the Debye Model):

(a)

$$\omega_D = \sqrt{\frac{3\pi^2 v^3 N}{V}},$$

where v is the speed of sound, N is the number of modes, and V is the volume of solid.

(b) Calculate the number of phonons N_{photons} in the system of temperature T.

(c) Show that at high temperature: $N_{\text{photons}} \propto T$.

(d) Show that at low temperature: $N_{\text{photons}} \propto T^3$.

Question 5

Consider Na metal at $30K$. Sodium has a $\rho = 970$ kg, and is monovalent. Speed of sound v in Na can be find by:

$$v = \sqrt{\frac{B}{\rho}},$$

where $B = 5.2 \times 10^9\ Nm^{-2}$ is the bulk modulus. What is the typical scattering angle θ when an electron scatters from a typical phonon?

Question 6

(a) Given the usual definitions $x = E/\hbar\omega_c$ and $x_0 = E_F/\hbar\omega_c$, show that the integrated density of states for electrons are:

$$\tilde{D}^{(0)}(x) = \frac{2}{3}(\hbar\omega_c)\frac{D(x_0)}{\sqrt{x_0}}x^{3/2},$$

$$\tilde{D}^{(B)}(x) = (\hbar\omega_c)\frac{D(x_0)}{\sqrt{x_0}}\sum_{\ell=0}^{\ell_{\max}}\sqrt{x - \left(\ell + \frac{1}{2}\right)}\;\Theta\left[x - \left(\ell + \frac{1}{2}\right)\right],$$

where Θ is the Heaviside function.

(b) Find the ratio $\frac{n^{(B)}}{n^{(0)}}$ where $n^{(B)}(n^{(0)})$ is the number of states upto the Fermi energy in the presence (absence) of a magnetic field. Plot $\frac{n^{(B)}}{n^{(0)}}$.

Question 7

(a) In class, we plotted δx (change in Fermi energy within magnetic field B) with respect to x_0. Plot δx with respect to B instead of x_0.

(b) Identify regions where only the lowest Landau levels (say $\ell = 0, 1, 2, 3$) are occupied.

(c) At what values of x_0, would we get the magnetic ground state ($\ell = 0$)?

Question 8

In class, we wrote down:

$$V(\vec{r}) = \sum_{\vec{G}} V_{\vec{G}}\, e^{i\vec{G}\cdot\vec{r}} \tag{1}$$

(a) Given

$$V(\vec{r}) = \frac{L^3}{(2\pi)^3} \int_{\text{all } \vec{k}} d^3k\ V(\vec{k}) e^{i\vec{k}\cdot\vec{r}}, \tag{2}$$

and using

$$\frac{1}{N} \sum_{\vec{R}} V(\vec{r} + \vec{R}) = V(\vec{r}), \tag{3}$$

derive Eq. (1).

(b) Show that if $V(\vec{r})$ is real, $V^*_{\vec{G}} = V_{-\vec{G}}$.

Question 9

(a) Derive the energy dispersion $E(k_x)$ in the vicinity of the zone edges, $k_x = \pm\frac{n\pi x}{a} + \delta$. Assume a weak periodic potential.

(b) What are the eigenfunctions of the perturbed Hamiltonian at the edge of the first Brillouin Zone?